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On the Second Approximation to the "Oseen" Solution for the Motion of a Viscous Fluid

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IV. *On the Second Approximation to the "Oseen" Solution for the Motion of a Viscous Fluid.*

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§ 1. *Statement of the Problem.*

IN a recent paper* the author obtained expressions for the forces on a stationary cylinder in a steady stream of incompressible viscous fluid and showed that the force transverse to the stream follows the well-known Kutta-Joukowski law, whereas the force in the direction of the stream itself is given by a similar law, involving, instead of the circulation, an outward radial flow, compensated by an intake along a " tail " behind the cylinder.

These results were obtained by considering the motion at a distance from the cylinder, and assuming that the velocities of disturbance from the uniform stream were so small that, at a sufficient distance, their squares and products could be neglected both in the equations of motion and in the integrals round a circle of large radius, in terms of which the forces on the cylinder were expressed.

The equations of motion, for steady motion, were substantially the same (though put into a somewhat different form) as those derived by OSEEN† for motion symmetrical about an axis and modified by LAMB‡ to suit the two-dimensional case. More recently, H. FAXÉN|| has dealt with the same equations and their solution, and his results confirm those obtained previously by the author.

In the paper referred to the author attempted, in § 9, to make some sort of estimate of the effect of the terms neglected. The method used was necessarily very imperfect, but a complete discussion of the second approximation would have involved considerable work and had to be postponed.

The main object of the present paper was to undertake a fuller investigation of the second order approximation, when the terms of second order in the velocities of disturbance are neglected neither in the equations of motion nor in the integrals giving the forces.

This fuller investigation was also made necessary by an attempt to obtain the torque, as well as the forces, on the cylinder. It appeared that terms of the second degree must

* " The Forces on a Cylinder in a Stream of Viscous Fluid," ' Roy. Soc. Proc.,' A, vol. 113, pp. 7–27 (1926).

† ' Arkiv för Matematik,' vol. 6, No. 29 (1910).

‡ ' Hydrodynamics ' (4th Ed.), § 343.

|| " Exakte Lösung der Oseenchen Differentialgleichungen einer Zähen Flüssigkeit für den Fall der Translationsbewegung eines Zylinders," ' Nova Acta R. Soc. Sci., Upsala ' (1927).

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be retained in the torque, even when we retain only terms of the first order in the solution of the equations, and it became essential to discover whether, if terms of the second order were retained in the solution of the equations, such terms would affect the value of the torque.

§ 2. *Brief Summary of Method and Results.*

The method adopted for solving the equations was the standard one, to substitute the approximate (first order) solution in the terms of second order in the equations; from this the unknowns may be obtained. This is done in §§ 4–11 of the present paper. The work is complicated by the necessity of satisfying certain conditions of continuity and one-valuedness.

The general solution of the equations of motion of the liquid having been obtained, exact expressions for the forces and torque are written down, in terms of integrals round a circle of very large radius, enclosing the cylinder. The contributions of the terms of the various orders to the forces and torque respectively are then examined in detail. It is found that the expressions for the forces, obtained in the paper referred to, still hold good, although the method of dealing with the terms neglected, attempted in that paper, turns out not to have been really valid, on account of the rise in importance of some second order terms on integration.

In the case of the torque, however, a very remarkable result emerges. One of the second order terms is found to lead to an integral which, instead of vanishing or tending to a finite limit when the radius R of the circle of integration is made infinite, actually becomes large of the order log R, a result which is clearly physically impossible, since the torque must have a definite finite value.

An alternative method of solution for this particular term is found to confirm the result.

It seems unlikely that terms of higher order can possibly cancel this term out, for the solution involves an infinite number of arbitrary constants, and it is difficult to see how algebraic expressions of the third and higher degrees in these can possibly cancel an expression of the second degree for *all possible* values of the constants.

Only two conclusions appear possible.

Owing to the difficulty in dealing with non-linear forms it has not here been possible, as was done in the previous paper, to deal with motion which is steady *only on the average*. The equations have therefore dealt throughout with rigorously steady motion.

It may be that such rigorously steady motion is impossible, in which case we should have met a difficulty analogous to STOKES' well-known paradox in the case of two-dimensional slow motion.

It has, however, been found that if we actually modify the equations suitably (*e.g.*, in the Oseen manner) STOKES' paradox disappears, so far as the forces are concerned. There appears no reason to suppose that the present paradox cannot be removed in an analogous manner.

This brings us to the alternative conclusion, namely, that even the Oseen form of the equations is unsuitable as a first approximation, upon which to base further developments, but that successive approximations should be based upon a different form.

An analogy from the Lunar Theory suggests itself. It is well known that, in this theory, the ordinary Keplerian ellipse cannot be used as the basis of successive approximations, but the first approximation which has to be substituted into the equations must take account of the motion of the lunar perigee, that is, must be a rotating Keplerian ellipse.

It is suggested that something of the same kind occurs in these hydrodynamic equations.

Looking at the matter from this point of view, it does not necessarily invalidate the Oseen type of solution, as giving an approximate picture of what really occurs, precisely as the Keplerian ellipse will give, for one revolution, a fair representation of the moon's motion.

All that is here shown is that such solutions cannot be used as the basis of further developments on the normal lines of successive approximations.

Finally, a type of equation is suggested which might afford such a basis. The equation is linear, but belongs to a class which does not seem to have been studied in detail.

§ 3. *Notation and Recapitulation of Previous Results.*

For brevity of reference the author's previous paper on the same subject (' Roy. Soc. Proc.,' A, vol. 113, pp. 7–27 (1926)) will be referred to as " Forces."

The fluid is taken throughout as incompressible.

The axes Ox, Oy of co-ordinates are taken parallel and perpendicular to the stream, the origin being anywhere in or near the cylindrical obstacle.

The resolutes of total velocity are denoted as usual by u and v; u' and v' are the velocities of disturbance, so that, if U is the stream-velocity at infinity

$$\left.\begin{aligned} u &= \mathrm{U} + u' \\ v &= v' \end{aligned}\right\} \qquad (3.01)$$

Suffixes r and θ are used to denote velocity resolutes in the radial and cross-radial directions; so that (u_r, u_θ) and (u_r', u_θ') are the radial and cross-radial resolutes of (u, v) and (u', v'), respectively.

ζ and Q denote the vorticity and total head respectively, so that

$$2\zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \qquad (3.02)$$

$$\mathrm{Q} = \frac{p}{\rho} + \Omega + \tfrac{1}{2}(u^2 + v^2), \qquad (3.03)$$

p being the pressure, ρ the density and Ω the body force potential. If Π is the hydrostatic pressure in the undisturbed stream,

$$Q' = \frac{p - \Pi}{\rho} + \tfrac{1}{2}(u^2 + v^2) - \tfrac{1}{2}U^2, \quad \ldots \ldots \quad (3.04)$$

$$\zeta' = \zeta. \quad \ldots \ldots \quad (3.05)$$

The equations of steady motion may be written in the form

$$\frac{\partial Q}{\partial x} + 2\nu \frac{\partial \zeta}{\partial y} = 2v\zeta, \quad \ldots \ldots \quad (3.06)$$

$$\frac{\partial Q}{\partial y} - 2\nu \frac{\partial \zeta}{\partial x} = -2u\zeta, \quad \ldots \ldots \quad (3.07)$$

and, since $\Pi/\rho + \Omega =$ const., these may be written in the form

$$\frac{\partial Q'}{\partial x} + 2\nu \frac{\partial \zeta'}{\partial y} = 2v'\zeta', \quad \ldots \ldots \quad (3.1)$$

$$\frac{\partial Q'}{\partial y} - 2\nu \frac{\partial \zeta'}{\partial x} + 2U\zeta' = -2u'\zeta'. \quad \ldots \ldots \quad (3.2)$$

In addition, we have a stream function ψ', derived from the equation of continuity

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0.$$

We now write

$$u' = u_0 + u_1 + u_2$$

$$v' = v_0 + v_1 + v_2,$$

where

(u_0, v_0) refer to a purely irrotational motion;

(u_1, v_1) refer to the rotational motion such that $u_0 + u_1$, $v_0 + v_1$ satisfy equations (3.1), (3.2) with the right-hand sides put equal to zero. These will be referred to as the *first order* solutions.

(u_2, v_2) refer to the motion which has to be superimposed upon $(u_0 + u_1, v_0 + v_1)$ to satisfy equations (3.1), (3.2), when the values derived from $u_0 + u_1$, $v_0 + v_1$ are substituted into the product terms on the right-hand side. These will be referred to as the *second order* solutions.

$\zeta_0, \zeta_1, \zeta_2$ are the vorticities derived from the above, so that $\zeta_0 = 0$.

Similarly, we can break up Q′ into the form $Q_0 + Q_1 + Q_2$, where, from (3.1), (3.2), we obtain

$$\frac{\partial Q_0}{\partial x} = 0, \qquad \frac{\partial Q_0}{\partial y} = 0, \quad \ldots \ldots \quad (3.31)$$

$$\frac{\partial Q_1}{\partial x} + 2\nu \frac{\partial \zeta_1}{\partial y} = 0, \qquad \frac{\partial Q_1}{\partial y} - 2\nu \frac{\partial \zeta_1}{\partial x} + 2U\zeta_1 = 0, \quad \ldots \quad (3.32)$$

$$\frac{\partial Q_2}{\partial x} + 2\nu \frac{\partial \zeta_2}{\partial y} = 2(v_0 + v_1)\zeta_1, \qquad \frac{\partial Q_2}{\partial y} - 2\nu \frac{\partial \zeta_2}{\partial x} + 2U\zeta_2 = -2(u_0 + u_1)\zeta_1, \quad (3.33)$$

so that $Q_0 = \text{const.}$

The corresponding stream functions ψ_0, ψ_1, ψ_2, satisfy the following equations $\left(\text{since } v = \frac{\partial \psi}{\partial x},\ u = -\frac{\partial \psi}{\partial y}\right)$

$$\nabla^2 \psi_0 = 0 \quad \ldots \ldots \quad (3.41)$$

$$\nabla^2 \psi_1 = 2\zeta_1 \quad \ldots \ldots \quad (3.42)$$

$$\nabla^2 \psi_2 = 2\zeta_2. \quad \ldots \ldots \quad (3.43)$$

Eliminating Q_1 and Q_2, equations (3.32) and (3.33) lead to the following equations for the vorticities

$$\nu \nabla^2 \zeta_1 - U\,\partial\zeta_1/\partial x = 0. \quad \ldots \ldots \quad (3.52)$$

$$\nu \nabla^2 \zeta_2 - U\,\partial\zeta_2/\partial x = \frac{\partial}{\partial x}\{(u_0 + u_1)\zeta_1\} + \frac{\partial}{\partial y}\{(v_0 + v_1)\zeta_1\}, \quad \ldots \quad (3.53)$$

$$= (u_0 + u_1)\frac{\partial \zeta_1}{\partial x} + (v_0 + v_1)\frac{\partial \zeta_1}{\partial y}, \quad \ldots \ldots \quad (3.531)$$

in virtue of the equation of continuity.

The solution of equations (3.42) and (3.52) is given in "Forces" and the velocities are found to be of the form*

$$\left.\begin{aligned} u_0 &= -\frac{\alpha_0}{\kappa^2}\frac{\cos\theta}{r} + \frac{d_0}{r}\sin\theta + \frac{c_1\cos 2\theta + d_1 \sin 2\theta}{r^2} + \frac{2(c_2\cos 3\theta + d_2\sin 3\theta)}{r^3} + \ldots \\ v_0 &= -\frac{\alpha_0}{\kappa^2}\frac{\sin\theta}{r} - \frac{d_0}{r}\cos\theta + \frac{c_1\sin 2\theta - d_1\cos 2\theta}{r^2} + \frac{2(c_2\sin 3\theta - d_2\cos 3\theta)}{r^3} + \ldots \end{aligned}\right\}, \quad (3.6)$$

$$u_1 = \frac{e^{\kappa x}}{\kappa}\sum_{n=0}^{\infty}\left[a_n(C_n + C_{n+1}) + b_n(S_n + S_{n+1})\right], \quad \ldots \ldots \quad (3.71)$$

$$v_1 = \frac{e^{\kappa x}}{\kappa}\sum_{n=0}^{\infty}\left[a_n(S_{n+1} - S_n) + b_n(C_n - C_{n+1})\right], \quad \ldots \ldots \quad (3.72)$$

* The notation has been slightly changed. Thus the constants here called a_n, b_n are denoted by $-\beta_n$, α_n in "Forces," what is here called α_0 is called $-\delta_0$ in "Forces," and the constants c_n, d_n are termed $-B_n/\kappa^2$, A_n/κ^2 respectively. From equations (59), (60) and (46) of "Forces" and the changes of notation here given, equations (3.6), (3.71) and (3.72) are easily deduced.

where $C_n = K_n(z)\cos n\theta$, $S_n = K_n(z)\sin n\theta$, and $z = \kappa r$, where $\kappa = U/2\nu$, and K_n is the Bessel function of the second class with imaginary argument.

The constants a_n, b_n, c_n, d_n are entirely arbitrary, but owing to certain conditions of one-valuedness satisfied by the stream function, the constant α_0 is related to the other constants of the solution by the relation

$$\alpha_0 = \sum_{n=0}^{\infty} a_n.$$

The vorticity ζ_1 is given by

$$\zeta_1 = \kappa v_1 = e^{\kappa x} \sum_{n=0}^{\infty} [a_n (S_{n+1} - S_n) + b_n (C_n - C_{n+1})]. \quad (3.73)$$

Equations (30) of "Forces" give certain formulæ which will be found useful in the reductions, namely,

$$(\partial C_n/\partial x) = -\tfrac{1}{2}\kappa (C_{n-1} + C_{n+1}), \quad (3.81)$$

$$(\partial C_n/\partial y) = \tfrac{1}{2}\kappa (S_{n-1} - S_{n+1}), \quad (3.82)$$

$$(\partial S_n/\partial x) = -\tfrac{1}{2}\kappa (S_{n-1} + S_{n+1}), \quad (3.83)$$

$$(\partial S_n/\partial y) = -\tfrac{1}{2}\kappa (C_{n-1} - C_{n+1}). \quad (3.84)$$

§ 4. *Transformation to Parabolic Co-Ordinates.*

If we bear in mind that when r is large, $K_n(\kappa r)$ approximates to the value

$$e^{-\kappa r}(\pi/2\kappa r)^{\frac{1}{2}},$$

it is clear that the right-hand side of (3.531) will contain factors $e^{\kappa(x-r)}$ and $e^{2\kappa(x-r)}$, and in order to ascertain the correct order of magnitudes of terms, arranging merely by powers of r is inadequate, since these exponential factors exercise a dominating effect. Further, since $\exp.\{\kappa(x-r)\} = \exp.\{-2\kappa r \sin^2 \tfrac{1}{2}\theta\}$, it is clear that, when r is large, this factor practically annihilates everything it multiplies, unless θ is of order less than $1/\sqrt{r}$. It will appear, as a consequence of this, that the form of an expression in θ will exert a very material influence on its order of magnitude, θ having to be reckoned as small of the order $1/\sqrt{r}$: similar considerations apply to the differential $d\theta$. It is largely on account of neglect of this consideration that the argument in §9 of "Forces" is inadequate.

In order to disentangle properly the orders of the terms, it was found desirable to express the exponential factors solely in terms of one co-ordinate.

Curvilinear co-ordinates were therefore employed in the solution, defined as follows:—

$$\xi + i\eta = 2\sqrt{\kappa}\,(x + iy)^{\frac{1}{2}}, \quad (4.0)$$

so that

$$\xi = 2z^{\frac{1}{2}} \cos \tfrac{1}{2}\theta, \quad (4.11)$$

$$\eta = 2z^{\frac{1}{2}} \sin \tfrac{1}{2}\theta, \quad (4.12)$$

and

$$z = \tfrac{1}{4}(\xi^2 + \eta^2), \qquad \theta = 2\tan^{-1}\left(\frac{\eta}{\xi}\right). \quad (4.13)$$

We have also

$$r + x = \xi^2/2\kappa, \quad \ldots \quad (4.21)$$

$$r - x = \eta^2/2\kappa, \quad \ldots \quad (4.22)$$

so that the exponential factors in question take the form $e^{-\frac{1}{2}\eta^2}$, $e^{-\eta^2}$.

In order to cover the plane a convention has to be introduced as to the ranges of ξ and η. We shall take

$$0 < \xi < \infty,$$

$$-\infty < \eta < +\infty.$$

This will necessitate investigation of what happens when $\xi = 0$, when discontinuities would generally be introduced. As, however, we shall deal solely with the region far away from the origin, ξ cannot be zero without η being large, and, save in exceptional cases which will be discussed, the exponential factors will of themselves ensure that all continuity conditions are satisfied. In fact, the right-hand sides of equation (3.531) will only be sensible in the "tail" behind the obstacle, where ξ will be very large and η will usually be finite. It is only in the process of adjusting discontinuities in this tail that solutions which remain finite elsewhere may make their appearance.

The transformations of the space-differential coefficients are put down here for reference.

$$\partial/\partial x = (2r)^{-1}[\xi\,\partial/\partial\xi - \eta\,\partial/\partial\eta], \quad \ldots \quad (4.31)$$

$$\partial/\partial y = (2r)^{-1}[\eta\,\partial/\partial\xi + \xi\,\partial/\partial\eta], \quad \ldots \quad (4.32)$$

$$\partial/\partial r = (2r)^{-1}[\xi\,\partial/\partial\xi + \eta\,\partial/\partial\eta], \quad \ldots \quad (4.33)$$

$$\partial/r\,\partial\theta = (2r)^{-1}[-\eta\,\partial/\partial\xi + \xi\,\partial/\partial\eta]. \quad \ldots \quad (4.34)$$

When the equations (3.43), (3.531) are transformed to these co-ordinates in the usual manner, they are found to become

$$\nu\kappa r^{-1}[\nabla_{\xi\eta}{}^2\zeta_2 - \xi\,\partial\zeta_2/\partial\xi + \eta\,\partial\zeta_2/\partial\eta] = (u_0 + u_1)\,\partial\zeta_1/\partial x + (v_0 + v_1)\,\partial\zeta_1/\partial y, \quad (4.4)$$

$$\kappa r^{-1}\nabla_{\xi\eta}{}^2\psi_2 = 2\zeta_2, \quad \ldots \quad (4.5)$$

where

$$\nabla_{\xi\eta}{}^2 \equiv \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2. \quad \ldots \quad (4.6)$$

§ 5. *Approximations for* $(u_0 + u_1)\,\partial\zeta_1/\partial x + (v_0 + v_1)\,\partial\zeta_1/\partial y$.

Using the asymptotic expansion for $K_n(z)$, namely,

$$K_n(z) = e^{-z}\sqrt{\frac{\pi}{2z}}\left[1 + \frac{4n^2 - 1}{1!\,8z} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2!\,(8z)^2} + \ldots\right], \quad . \quad (5.0)$$

we obtain, after some fairly long but straightforward reductions, the following approximate expressions when ξ is large compared with η :—

$$e^{z}\,.\,C_n = \sqrt{2\pi}\left[\frac{1}{\xi}+\frac{1}{\xi^3}\{2(n^2-\tfrac{1}{4})-2\eta^2(n^2+\tfrac{1}{4})\} + \frac{1}{\xi^5}\left\{\begin{array}{c}2(n^2-\tfrac{1}{4})(n^2-\tfrac{9}{4})-4\eta^2(n^2-\tfrac{1}{4})(n^2+\tfrac{3}{4})\\ +\frac{\eta^4}{6}(4n^4+14n^2+\tfrac{9}{4})\end{array}\right\}\right], \quad (5.11)$$

$$e^{z}\,.\,S_n = \sqrt{2\pi}\left[\frac{2n\eta}{\xi^2}+\frac{n}{\xi^4}\{(4n^2-1)\eta-(4n^2+5)\eta^3/3\} + \frac{n}{\xi^6}\left\{\begin{array}{c}(4n^2-1)(4n^2-9)\eta/4-(4n^2+11)(4n^2-1)\eta^3/6\\ +(16n^4+120n^2+89)\eta^5/60\end{array}\right\}\right], \quad (5.12)$$

if we neglect powers of $1/\xi$ greater than the sixth.

Applying equations (3.81) to (3.84) we obtain

$$\frac{\partial\zeta_1}{\partial x} = \kappa e^{\kappa x}\,\Sigma\,\{a_n(S_{n+1}-S_n)+b_n(C_n-C_{n+1})\}$$
$$-\tfrac{1}{2}\kappa e^{\kappa x}\,\Sigma\,\{a_n(S_{n+2}-S_{n+1}+S_n-S_{n-1})+b_n(C_{n-1}-C_n+C_{n+1}-C_{n+2})\}, \quad (5.13)$$

$$\frac{\partial\zeta_1}{\partial y} = -\tfrac{1}{2}\kappa e^{\kappa x}\,\Sigma\,\{a_n(C_n-C_{n-1}+C_{n+1}-C_{x+2})+b_n(S_{n+1}-S_{n+2}-S_{n-1}+S_n)\}. \quad (5.14)$$

Substituting into these from (5.11) and (5.12) we have

$$\frac{\partial\zeta_1}{\partial x} = \kappa\,\sqrt{2\pi}e^{-\eta^2/2}\left[\frac{\alpha_0}{\xi^4}(4\eta^3-12\eta)+\frac{\beta}{\xi^5}(4\eta^4-24\eta^2+12)\right], \quad (5.21)$$

$$\frac{\partial\zeta_1}{\partial y} = \kappa\,\sqrt{2\pi}e^{-\eta^2/2}\left[\frac{4\alpha_0(1-\eta^2)}{\xi^3}+\frac{\beta}{\xi^4}(12\eta-4\eta^3)\right], \quad (5.22)$$

where

$$\beta = \Sigma\,(2n+1)\,b_n. \quad (5.23)$$

In the above only the two most important terms in $1/\xi$ have been retained in each case.

Treating u_0+u_1 and v_0+v_1 in the same manner, we derive the approximate expressions

$$u_0+u_1 = -\frac{4\alpha_0}{\kappa}\frac{1}{\xi^2}+\frac{\sqrt{2\pi}}{\kappa}e^{-\eta^2/2}\left[\frac{2\alpha_0}{\xi}+\frac{2\beta\eta}{\xi^2}\right], \quad (5.31)$$

$$v_0+v_1 = -\frac{4d_0\kappa}{\xi^2}-\frac{8\alpha_0\eta}{\kappa\xi^3}+\frac{\sqrt{2\pi}}{\kappa}e^{-\eta^2/2}\left[\frac{2\alpha_0\eta}{\xi^2}-\frac{2\beta(1-\eta^2)}{\xi^3}\right], \quad (5.32)$$

whence, finally, denoting for shortness the expression

$$(u_0+u_1)\,\partial\zeta_1/\partial x+(v_0+v_1)\,\partial\zeta_1/\partial x$$

by R, we arrive at the result

$$R = R_1(\eta) \,.\, \xi^{-5} + R_2(\eta) \,.\, \xi^{-6}, \quad \dots \dots \dots \quad (5.40)$$

where

$$R_1 = L_1 (1 - \eta^2)\, e^{-\eta^2/2} + M_1 \eta e^{-\eta^2}, \quad \dots \dots \dots \quad (5.41)$$

$$R_2 = L_2 (\eta + \eta^3)\, e^{-\eta^2/2} + M_2\, e^{-\eta^2/2}(3\eta - \eta^3) + N_2 (1 - 2\eta^2)\, e^{-\eta^2}, \quad \dots \quad (5.42)$$

L_1, M_1, L_2, M_2, N_2 being constants given by the equations

$$\left.\begin{array}{l} L_1 = -16\sqrt{2\pi}\alpha_0 d_0 \kappa^2, \qquad M_1 = -32\pi\alpha_0^2, \\ L_2 = 16\sqrt{2\pi}\alpha_0^2, \quad M_2 = -16\sqrt{2\pi}\,\beta d_0 \kappa^2, \quad N_2 = 32\pi\alpha_0\beta \end{array}\right\} \quad \dots \quad (5.43)$$

§ 6. *The Integration of the Equation for* $\zeta_2 - \kappa v_2$.

The equation (3.531) may be written

$$\nu (\nabla^2 - 2\kappa\, \partial/\partial x)\, \zeta_2 = R,$$

or, using (3.43),

$$\nu (\nabla^2 - 2\kappa\, \partial/\partial x)\, \nabla^2 \psi_2 = 2R.$$

The two differential operators are clearly interchangeable, so that we may write the above

$$\nu\nabla^2 \,.\, \left(\nabla^2\psi_2 - 2\kappa \frac{\partial \psi_2}{\partial x}\right) = 2R,$$

or

$$\nu\nabla^2 \,.\, (\zeta_2 - \kappa v_2) = R. \quad \dots \dots \dots \dots \quad (6.0)$$

We proceed to obtain first of all the solution of this equation in the form

$$2\zeta_2 - 2\kappa v_2 = Z,$$

the equation for ψ_2 then becoming

$$\nabla^2 \psi_2 - 2\kappa \frac{\partial \psi_2}{\partial x} = Z. \quad \dots \dots \dots \dots \quad (6.1)$$

We note incidentally that all we require is to obtain (subject to the conditions of continuity) particular integrals of the above equations; for it is clear that any solutions of the equation

$$\nabla^2 \left(\nabla^2 - 2\kappa \frac{\partial}{\partial x}\right) \psi = 0$$

are included in ψ_0 and ψ_1, the most general solutions of which (satisfying the conditions of continuity) have already been obtained.

For our present purpose it will be sufficient to restrict R to its leading term; the terms arising from the second term are found to be negligible in comparison. We thus obtain the approximate equation

$$\frac{\kappa\nu}{r} \nabla_{\xi\eta}^2 (\zeta_2 - \kappa v_2) = R_1(\eta)/\xi^5,$$

or, to the same approximation,

$$\nabla_{\xi\eta}^2 (2\zeta_2 - 2\kappa v_2) = \frac{1}{\kappa U} \frac{\{L_1 (1 - \eta^2) e^{-\eta^2/2} + M_1 \eta e^{-\eta^2}\}}{\xi^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.2)$$

Assume for $2\zeta_2 - 2\kappa v_2$ a solution in descending powers of ξ

$$2\zeta_2 - 2\kappa v_2 = Z = \frac{Z_3(\eta)}{\xi^3} + \frac{Z_4(\eta)}{\xi^4} + \dots \dots \dots \quad (6.3)$$

Substituting and equating coefficients of $1/\xi^3$

$$\frac{d^2 Z_3}{d\eta^2} = (L_1/\kappa U)(1 - \eta^2) e^{-\eta^2/2} + (M_1/\kappa U) \eta e^{-\eta^2},$$

leading to

$$Z_3 = -(L_1/\kappa U) e^{-\eta^2/2} + (M_1/2\kappa U) \int_\eta^\infty e^{-\eta^2} d\eta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.4)$$

This make Z_3 exponentially small when η is large and positive, but if η is large and negative it gives

$$Z_3 = (M_1 \sqrt{\pi}/2\kappa U),$$

leading to $\zeta_2 - \kappa v_2$ of order $1/r^{3/2}$, and not exponentially small.

Let us now denote by $E(x)$ a function defined as follows :—

$$\left. \begin{aligned} E(x) &= \int_x^\infty e^{-x^2} dx \qquad \text{if} \quad x > 0 \\ E(x) &= \int_x^{-\infty} e^{-x^2} dx \qquad \text{if} \quad x < 0 \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.5)$$

Then $E(x)$ has a discontinuity from $-\frac{1}{2}\sqrt{\pi}$ to $+\frac{1}{2}\sqrt{\pi}$ when passing through zero, but $dE(x)/dx = -e^{-x^2}$ and is entirely continuous. $E(x)$ is clearly an odd function.

If we then write

$$Z_3 = -(L_1/\kappa U) e^{-\eta^2/2} + (M_1/2\kappa U) E(\eta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.6)$$

Z_3 is throughout exponentially small when η is large, either positively or negatively.

On the other hand, we have introduced into Z_3 a discontinuity when $\eta = 0$, which will have to be taken into account.

§ 7. *The Integration for* ψ_2.

Before discussing further the discontinuity introduced into the M_1-term, it will be convenient to carry out the integration of equation (6.1).

This equation, when transformed to the parabolic co-ordinates ξ, η, takes the form

$$\frac{k}{r} [\nabla_{\xi\eta}^2 \psi_2 - \xi \, \partial\psi_2/\partial\xi + \eta \, \partial\psi_2/\partial\eta] = Z_3/\xi^3 + \dots,$$

or, to the approximation used,

$$\nabla_{\xi\eta}^2\psi_2 - \xi\,\partial\psi_2/\partial\xi + \eta\,\partial\psi_2/\partial\eta = Z_3/4\kappa^2\xi + \ldots \qquad (7.01)$$

Assuming

$$\psi_2 = \Psi_1/\xi + \ldots \qquad (7.02)$$

the equation satisfied by Ψ_1 is found to be

$$\frac{d^2\Psi_1}{d\eta^2} + \Psi_1 + \eta\frac{d\Psi_1}{d\eta} = Z_3/4\kappa^2. \qquad (7.03)$$

The left-hand side being a perfect differential coefficient as it stands, the equation is solved by the usual process and the solution is found to be

$$\Psi_1 = e^{-\eta^2/2}\int e^{\eta^2/2}\left\{\int (Z_3/4\kappa^2)\,d\eta\right\}d\eta.$$

Substituting for Z_3 from (6.6) we note that

$$\int e^{-\eta^2/2}\,d\eta = -\sqrt{2}\,E\left(\frac{\eta}{\sqrt{2}}\right), \qquad (7.1)$$

$$\int E(\eta)\,d\eta = \eta\,E(\eta) - \tfrac{1}{2}e^{-\eta^2}, \qquad 7.2)$$

and, further,

$$\begin{aligned}\int e^{\eta^2/2}\left(\eta\,E(\eta) - \tfrac{1}{2}e^{-\eta^2}\right)d\eta &= \int \eta\,e^{\eta^2/2}\,E(\eta)\,d\eta - \tfrac{1}{2}\int e^{-\eta^2/2}\,d\eta \\ &= e^{\eta^2/2}\,E(\eta) - \int e^{\eta^2/2}\,E'(\eta)\,d\eta - \tfrac{1}{2}\int e^{-\eta^2/2}\,d\eta \\ &= e^{\eta^2/2}\,E(\eta) + \tfrac{1}{2}\int e^{-\eta^2/2}\,d\eta, \qquad (7.3)\end{aligned}$$

and, since we only require particular integrals, we may, in such integrals as we wish, make the limits 0 and η. Thus

$$\Psi_1 = (L_1\sqrt{2}/4\kappa^3 U)\,e^{-\eta^2/2}\int_0^\eta e^{\eta^2/2}\,E\left(\frac{\eta}{\sqrt{2}}\right)d\eta + (M_1/8\kappa^3 U)\left[E(\eta) + \tfrac{1}{2}e^{-\eta^2/2}\int_0^\eta e^{-\eta^2/2}\,d\eta\right], \quad (7.4)$$

and

$$\begin{aligned}\psi_2 = {} & (L_1\sqrt{2}/4\kappa^3 U\xi)\,e^{-\eta^2/2}\int_0^\eta e^{\eta^2/2}\,E\left(\frac{\eta}{\sqrt{2}}\right)d\eta \\ & + (M_1/8\kappa^3 U\xi)\left[E(\eta) + \tfrac{1}{2}e^{-\eta^2 2}\int_0^\eta e^{-\eta^2/2}\,d\eta\right] \qquad (7.5)\end{aligned}$$

It may be shown that when η is large, $E(\eta/\sqrt{2})$ approximates to $e^{-\eta^2/2}/(\sqrt{2}\eta)$, so that $\int_0^\eta e^{\eta^2/2}\,E(\eta/\sqrt{2})\,d\eta$ approximates to $(\log\eta)/\sqrt{2}$ numerically. Hence the term multiplied by L_1 in (7.5) tends to become small exponentially. The same is readily seen to be true of the term multiplied by M_1.

In ψ_2 itself the M_1-term is the only one which introduces a discontinuity. This discontinuity is from $-M_1\sqrt{\pi}/16\kappa^3U\xi$ to $+M_1\sqrt{\pi}/16\kappa^3U\xi$ when η increases through 0.

In $\frac{\partial\psi_2}{\partial\eta}$, however, the M_1-term introduces no discontinuity, but the L_1-term contains a part

$$L_1\sqrt{2}\,.\,E(\eta/\sqrt{2})/4\kappa^3U\xi,$$

and this involves a discontinuity from $-L_1\sqrt{2\pi}/8\kappa^3U\xi$ to $+L_1\sqrt{2\pi}/8\kappa^3U\xi$ when η increases through 0.

In order to remove these discontinuities, "complementary function" solutions have to be introduced, so as to reduce our particular integrals to a suitable form. These complementary function solutions are of the types which have been denoted by suffixes 0 and 1, but have been excluded from the solutions given under these heads, because of these very discontinuities which they involve.

§ 8. *The Removal of the Discontinuity in the* M_1-*Term.*

We begin with the M_1-term, as the work is rather easier and the principles involved will appear more clearly.

If we consider an irrotational stream function of the type

$$\psi = A(\cos\tfrac{1}{2}\theta)/r^{\frac{1}{2}},$$

where θ is allowed to run from 0 to 2π, it appears that $(\psi)_{\theta=0} = A/r^{\frac{1}{2}} = 2\sqrt{\kappa}A/\xi$ approximately, whereas $(\psi)_{\theta=2\pi} = -2\sqrt{\kappa}A/\xi$ approximately.

If we make $A = M_1\sqrt{\pi}/32\kappa^{7/2}U$, these discontinuities are identical with those in the M_1-term of ψ_2, previously found. It is easily verified that there is no discontinuity in $\frac{\partial\psi}{\partial\theta}$, and therefore in $\frac{\partial\psi}{\partial\eta}$, and further, that there exists a discontinuity in $2\zeta - 2\kappa v$, *i.e.*, in $-2\kappa v$, of which the value is $\kappa A\cos\frac{3}{2}\theta/r^{3/2}$. This latter discontinuity is from

$$-8A\kappa^{5/2}/\xi^3 \quad \text{to} \quad +8A\kappa^{5/2}/\xi^3,$$

or, putting in the value of A just found, from

$$-M_1\sqrt{\pi}/4\kappa U\xi^3 \quad \text{to} \quad +M_1\sqrt{\pi}/4\kappa U\xi^3,$$

and this is exactly the discontinuity which we have found in the M_1-term of $2\zeta_2 - 2\kappa v_2$. Thus the discontinuities noted will be removed if we subtract such a solution from the one already found.

It is necessary, however, to make certain that the conditions of continuity are completely satisfied. Now these conditions, in a viscous fluid, involve not only the continuity of the velocity resolutes, but that of the stresses.

The latter are given by the equations—

$$\left.\begin{aligned}\widehat{xx} &= -p + 2\mu\frac{\partial u}{\partial x}\\ \widehat{yy} &= -p + 2\mu\frac{\partial v}{\partial y}\\ \widehat{xy} &= \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\end{aligned}\right\}, \quad \ldots \ldots \ldots \quad (8.0)$$

p being the hydrostatic pressure. We can take the line of discontinuity to be (as it actually is in this case) the axis of x.

From the continuity of ψ and $\frac{\partial \psi}{\partial y}$ follow the continuity of u and v, and therefore of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$. From the continuity of v and of $2\zeta - 2\kappa v$, follows the continuity of 2ζ. From the continuity of 2ζ and of $\frac{\partial v}{\partial x}$ follows (from the defining equation for ζ) the continuity of $\frac{\partial u}{\partial y}$. And from the continuity of $\frac{\partial u}{\partial x}$ and the equation $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ follows the continuity of $\frac{\partial v}{\partial y}$.

Finally, if we refer to the equations (3.06) and (3.07), provided $\partial\zeta/\partial y$ is also continuous, both $\frac{\partial Q}{\partial x}$ and $\frac{\partial Q}{\partial y}$ are continuous; hence Q, and therefore the pressure p, can be made continuous by adjusting the constant of integration, and all the conditions are satisfied, since every expression entering into the stresses has been shown to be continuous.

We have therefore still to show that our combination makes $\partial\zeta/\partial y$ or, what amounts to the same thing, $\frac{\partial}{\partial y}(\zeta - \kappa v)$ continuous. This will follow if there is no discontinuity in $\frac{\partial}{\partial \eta}(\zeta - \kappa v)$ or in $\frac{\partial}{\partial \theta}(\zeta - \kappa v)$.

But it is immediately found that, in the case of the irrotational solution of the present section, $\frac{\partial}{\partial \theta}(\zeta - \kappa v)$ involves $\sin \frac{3}{2}\theta$ as factor and so cannot be discontinuous. On the other hand, if we examine the M_1-part of the solution of § 6, it is at once seen that

$$\frac{\partial}{\partial \eta}(2\zeta_2 - 2\kappa v_2) = -(M_1/2\kappa U)\, e^{-\eta^2},$$

which is continuous.

We may therefore assume, as a solution satisfying all conditions to the present approximation,

$$\psi_2 = (M_1/8\kappa^3 U)\left[\left\{E(\eta) + \tfrac{1}{2}e^{-\eta^2/2}\int_0^\eta e^{-\eta^2/2}\,d\eta\right\}\xi^{-1} - \sqrt{\pi}\cos\tfrac{1}{2}\theta/4\sqrt{\kappa r}\right], \quad . \quad (8.1)$$

where we have to be careful to take θ from 0 to 2π and not from $-\pi$ to $+\pi$.

It will be noticed that this solution does not become exponentially small as we go away from the tail. It will introduce velocities of order $r^{-3/2}$, which are indeed smaller than the most important terms in u_0, v_0, but much more important than the terms in u_1, v_1.

§ 9. *The Removal of the Discontinuity in the L_1-Term.*

It is found that no irrotational stream function will remove in the same way the discontinuity in the L_1-term. We have to look for a stream function ψ which will have no discontinuity, but for which $\frac{\partial\psi}{\partial\eta}$ has a discontinuity of order $1/\xi$. Since, from equation (4.34) we have, when $\eta = 0$, $\frac{1}{2}\xi\frac{\partial}{\partial\eta} = \frac{\partial}{\partial\theta}$, it follows that the function ψ required should be such that $\frac{\partial\psi}{\partial\theta}$ has a finite discontinuity at $\theta = 0, 2\pi$.

A function which has a finite discontinuity and which satisfies the equation

$$\nabla^2\psi - 2\kappa\frac{\partial\psi}{\partial x} = 0$$

is given by equation (48) in "Forces." It is

$$\int_0^\theta z(K_0\cos\chi + K_1)e^{z\cos\chi}\,d\chi. \qquad (9.0)$$

In order to get the discontinuity in the form of a reversal of sign at $\theta = 0$, it is desirable to take the lower limit of the integral as π. This merely subtracts a constant from the integral, since

$$\int_0^\pi z(K_0\cos\chi + K_1)e^{z\cos\chi}\,d\chi = \pi z(K_0I_1 + K_1I_0) = \pi,$$

by a well-known result in Bessel functions.

Let us write for shortness

$$f(\theta, z) = \int_\pi^\theta z(K_0\cos\theta + K_1)e^{z\cos\theta}\,d\theta. \qquad (9.1)$$

Since f is known to be a solution of the equation

$$\nabla^2 f - 2\kappa\,\partial f/\partial x = 0;$$

therefore if

$$G = e^{-\kappa x}.f = e^{-z\cos\theta}f, \qquad (9.2)$$

G satisfies the equation

$$(\nabla^2 - \kappa^2)G = 0$$

or

$$\left(\frac{\partial^2}{\partial z^2} + \frac{1}{z}\frac{\partial}{\partial z} + \frac{1}{z^2}\frac{\partial^2}{\partial\theta^2} - 1\right)G = 0. \qquad (9.3)$$

Substitute now into the above equation F instead of G, where

$$F = \int_{\alpha}^{\theta} G \, d\theta. \qquad (9.4)$$

We find easily

$$\frac{\partial^2 F}{\partial z^2} + \frac{1}{z}\frac{\partial F}{\partial z} + \frac{1}{z^2}\frac{\partial^2 F}{\partial \theta^2} - F = \frac{1}{z^2}\left(\frac{\partial G}{\partial \theta}\right)_{\theta=\alpha}. \qquad (9.5)$$

The right-hand side is a function of z only.

If, therefore, $F_0(z)$ is any particular integral of the equation

$$\frac{d^2F_0}{dz^2} + \frac{1}{z}\frac{dF_0}{dz} - F_0 = \frac{1}{z^2}\left(\frac{\partial G}{\partial \theta}\right)_{\theta=\alpha}, \qquad (9.6)$$

then $F - F_0$ is a solution of the equation

$$(\nabla^2 - \kappa^2)(F - F_0) = 0, \qquad (9.61)$$

and $e^{z\cos\theta}(F - F_0)$ is a solution of the equation

$$\nabla^2\psi - 2\kappa\frac{\partial\psi}{\partial x} = 0. \qquad (9.62)$$

Take $\alpha = 0$.

Now, G being defined by equation (9.2), we find that

$$\left(\frac{\partial G}{\partial \theta}\right)_{\theta=0} = z(K_0 + K_1) \qquad (9.71)$$

and the equation for F_0 is

$$\frac{d^2F_0}{dz^2} + \frac{1}{z}\frac{dF_0}{dz} - F_0 = \frac{K_0 + K_1}{z}. \qquad (9.72)$$

Thus

$$\psi_2 = e^{z\cos\theta}\int_0^{\theta} e^{-z\cos\theta} . f \, d\theta - F_0 e^{z\cos\theta} \qquad (9.73)$$

is a possible stream function. We have to show that it possesses the correct type of discontinuity.

The form of F_0 will be investigated more closely a little later. For the present it is sufficient to note that, F_0 being a function of z only, $F_0 e^{z\cos\theta}$ possesses no discontinuity in θ, nor do its differential coefficients of any order.

If in $f = \int_{\pi}^{\theta} z(K_0\cos\theta + K_1)e^{z\cos\theta}\,d\theta$, we write $\theta = 2\pi - \theta'$,

then

$$f(\theta) = -\int_{\pi}^{\theta'} z(K_0\cos\theta + K_1)e^{z\cos\theta}\,d\theta = -f(\theta').$$

Thus $f(2\pi - \theta) = -f(\theta)$, so that

$$\int_0^{2\pi} e^{-z\cos\theta} f(\theta)\, d\theta = 0$$

and $\psi_2(2\pi) = -F_0 e^z = \psi_2(0)$, so that ψ_2 itself is continuous. On the other hand,

$$\frac{\partial\psi_2}{\partial\theta} = -z\sin\theta\, e^{z\cos\theta}\int_0^\theta e^{-z\cos\theta} f\,.\,d\theta + f - \frac{\partial}{\partial\theta}(F_0 e^{z\cos\theta}). \quad . \quad . \quad . \quad . \quad (9.74)$$

The only term here which can introduce a discontinuity at $\theta = 2\pi$, 0 is the term f. This changes from $+\pi$ to $-\pi$ as we cross the axis of x in the sense of increasing θ.

Now, our previously found L_1-term in ψ_2 (see equation (7.5)) introduced in $\dfrac{\partial\psi_2}{\partial\eta}$ a discontinuity from $-L_1\sqrt{2\pi}/8\kappa^3 U\xi$ to $+L_1\sqrt{2\pi}/8\kappa^3 U\xi$, that is, in $\dfrac{\partial\psi_2}{\partial\theta}$ a discontinuity from $-L_1\sqrt{2\pi}/16\kappa^3 U$ to $L_1\sqrt{2\pi}/16\kappa^3 U$. Thus the expression

$$\psi_2 = \frac{L_1\sqrt{2}}{4\kappa^3 U}\left[e^{-\eta^2/2}\,\xi^{-1}\int_0^\eta e^{\eta^2/2}\, E\left(\frac{\eta}{\sqrt{2}}\right)d\eta + \frac{e^{z\cos\theta}}{4\sqrt{\pi}}\left\{\int_0^\theta e^{-z\cos\theta} f d\theta - F_0\right\}\right] \quad . \quad (9.8)$$

gives no discontinuity at $\eta = 0$ for either ψ_2 or $\dfrac{\partial\psi_2}{\partial\eta}$.

We have still to show that $\zeta_2 - \kappa v_2$ and $\dfrac{\partial}{\partial\eta}(\zeta_2 - \kappa v_2)$ are continuous. Reference to the results of § 6 shows at once that this is true of the first term in (9.8), and since the second term is known to be a solution of

$$\nabla^2\psi_2 - 2\kappa\frac{\partial\psi_2}{\partial x} = 0,$$

i.e.,

$$\zeta_2 - \kappa v_2 = 0,$$

it must necessarily hold good of the second term also.

Accordingly, the value of ψ_2 in (9.8) is an integral of our equations which satisfies all the conditions of continuity, so that the complete form for ψ_2 is obtained by addition of (9.8) and (8.1) and gives

$$\psi_2 = (L_1\sqrt{2}/4\kappa^3 U)\left[e^{-\eta^2/2}\,\xi^{-1}\int_0^\eta e^{\eta^2/2}\, E(\eta/\sqrt{2})\, d\eta + (e^{z\cos\theta}/4\sqrt{\pi})\left\{\int_0^\theta e^{-z\cos 0} f d\theta - F_0\right\}\right]$$
$$+ (M_1/8\kappa^3 U)\left[\left\{E(\eta) + \tfrac{1}{2}e^{-\eta^2/2}\int_0^\eta e^{-\eta^2/2}\, d\eta\right\}\xi^{-1} - \sqrt{\pi}\cos\tfrac{1}{2}\theta/4\sqrt{\kappa r}\right], \quad (9.9)$$

whence we find, in a straightforward manner,

$$\zeta_2 - \kappa v_2 = -(L_1/2\kappa U)e^{-\eta^2/2}\,\xi^{-3} + (M_1/4\kappa U)\left[E(\eta)/\xi^3 - (\sqrt{\pi}/16)\cos\tfrac{3}{2}\theta/\kappa^{3/2} r^{3/2}\right]. \ (9.91)$$

§ 10. *The Form of the Function* F_0.

The form of the function F_0 exercises an important influence on the final result.

If we revert to the equation (9.6) which is satisfied by F_0 and write $F_0 = - K_0 \log z + F_1$, we find easily that

$$\left(\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} - 1\right) K_0 \log z = -\frac{2K_1}{z},$$

so that

$$\frac{d^2F_1}{dz^2} + \frac{1}{z}\frac{dF_1}{dz} - F_1 = \frac{K_0 - K_1}{z}, \quad \dots\dots\dots\dots \quad (10.1)$$

and, substituting the asymptotic expressions for K_0, K_1 (*cf.* equation (5.0)), we have

$$\frac{K_0 - K_1}{z} = e^{-z}\sqrt{\pi/2}\,(-\tfrac{1}{2}z^{-5/2} + \tfrac{3}{16}z^{-7/2} + \dots),$$

when z is large.

Assume for F_1 the form

$$F_1 = e^{-z}\sqrt{\pi/2}\,(A_0 z^{-3/2} + A_1 z^{-5/2} + \dots).$$

Substitute into (10.1) and equate coefficients of powers of z, we obtain

$$2A_0 = -\tfrac{1}{2}$$

$$\tfrac{9}{4}A_0 + 4A_1 = \tfrac{3}{16},$$

etc.,

where

$$A_0 = -\tfrac{1}{4}, \quad A_1 = +\tfrac{3}{16}, \text{ etc.}$$

It should be noted that this procedure could not be adopted in the case of the original equation (9.6). For in this case the leading term would contain $e^{-z} . z^{-3/2}$, and if we assume for F_1 a series beginning with $e^{-z} z^{-\frac{1}{2}}$, it is found that the term on the left-hand side in $e^{-z} . z^{-3/2}$ vanishes identically, so that the equation cannot be satisfied in this way.

We thus obtain the approximation, when z is large,

$$F_0 = - K_0 \log z - \sqrt{\pi/2}\, e^{-z}(\tfrac{1}{4}z^{-3/2} - \tfrac{3}{16}z^{-5/2} + \dots). \quad \dots\dots \quad (10.2)$$

§ 11. *The Magnitude of the* L_1*-Term when* z *is Large.*

If we refer to the solution (9.8) for the part of the stream-function multiplied by L_1, the first term is of order $1/\xi$ and is exponentially small (in η^2) away from the tail.

To examine the magnitude of the second term, we note that, when z is large,

$$f = \sqrt{\pi z/2}\int_{\pi}^{\theta}(1 + \cos\theta)\, e^{z(\cos\theta - 1)}\, d\theta, \quad \dots\dots\dots\dots \quad (11.0)$$

approximately, so that

$$f = \sqrt{2\pi}\int_{\pi}^{\theta}\cos\frac{\theta}{2} . e^{-2z\sin^2\frac{\theta}{2}}\, d\left(2\sqrt{z}\sin\frac{\theta}{2}\right),$$

and, remembering that $2\sqrt{z}\sin\theta/2 = \eta$, and $\cos\theta/2 = \dfrac{\xi}{\sqrt{\xi^2+\eta^2}}$, this transforms to

$$f = \sqrt{2\pi}\int_{\sqrt{\xi^2+\eta^2}}^{\eta} \xi\,(\xi^2+\eta^2)^{-\frac{1}{2}}\,e^{-\eta^2/2}\,d\eta, \quad \ldots \ldots \quad (11.11)$$

where η is > 0, and $0 < \theta < \pi$.

On the other hand, if $\pi < \theta < 2\pi$, η is negative, and

$$2\sqrt{z}\sin\theta/2 = -\eta, \qquad \cos\theta/2 = -\xi/\sqrt{\xi^2+\eta^2},$$

so that

$$f = \sqrt{2\pi}\int_{-\sqrt{\xi^2+\eta^2}}^{\eta} \xi\,(\xi^2+\eta^2)^{-\frac{1}{2}}\,e^{-\eta^2/2}\,d\eta. \quad \ldots \ldots \quad (11.12)$$

From (11.11) and (11.12) we have

$$f = -2\sqrt{\pi}\,\mathrm{E}\left(\frac{\eta}{\sqrt{2}}\right) + \text{terms of order } 1/\xi^2. \quad \ldots \ldots \quad (11.2)$$

Thus

$$\begin{aligned} e^{z\cos\theta}\int_0^\theta e^{-z\cos\theta} f\,d\theta &= e^{z(\cos\theta-1)}\int_0^\theta e^{+z(1-\cos\theta)} f\,d\left(2\sqrt{z}\sin\frac{\theta}{2}\right)\Big/\left(\sqrt{z}\cos\frac{\theta}{2}\right) \\ &= e^{-\eta^2/2}\int_0^\eta 2e^{\eta^2/2}\,(f/\xi)\,d\eta \\ &= -4\sqrt{\pi}\left\{e^{-\eta^2/2}\int_0^\eta e^{\eta^2/2}\,\mathrm{E}\,(\eta/\sqrt{2})\,d\eta\right\}\Big/\xi + \text{terms of order } 1/\xi^3. \quad \ldots \quad (11.3) \end{aligned}$$

On substituting this into (9.8) it appears that, to the first approximation, this part of the added solution exactly cancels the first term in (9.8), so that, neglecting terms in ψ_2 of order $1/\xi^3$, the part of ψ_2 due to L_1 reduces to

$$\psi_2 = -(\mathrm{L}_1\sqrt{2/\pi}\,.\,\mathrm{F}_0 e^{z\cos\theta})/16\kappa^3\mathrm{U}. \quad \ldots \ldots \quad (11.4)$$

Now from (10.2) the series part of F_0 will lead to terms in ψ_2 of order

$$e^{z(\cos\theta-1)}\,.\,z^{-3/2},$$

that is, of order $e^{-\eta^2/2}/\xi^3$, which we have already neglected.

We thus arrive, finally, at the most important L_1-term in the form

$$\psi_2 = (\mathrm{L}_1\sqrt{2/\pi}\,\mathrm{K}_0\,.\,e^{z\cos\theta}\,.\,\log z)/16\kappa^3\mathrm{U}, \quad \ldots \ldots \quad (11.5)$$

which becomes, when expressed in terms of ξ and η

$$\psi_2 = (\mathrm{L}_1/4\kappa^3\mathrm{U})\,e^{-\eta^2/2}\,\{\log(\xi/2)\}/\xi, \quad \ldots \ldots \quad (11.6)$$

neglecting terms of order $1/\xi^3$ in ψ_2.

§ 12. *A First Alternative Method of obtaining the* L_1-*Term.*

The presence of the logarithmic term appearing in (11.6) turns out to be so fundamental that it is desirable to confirm it in other ways.

If we return to the equation (7.01) and consider only the part of Z_3 involving L_1, we have

$$\nabla_{\xi\eta}{}^2\psi_2 - \xi\frac{\partial\psi_2}{\partial\xi} + \eta\frac{\partial\psi_2}{\partial\eta} = -(L_1/4\kappa^3 U)\, e^{-\eta^2/2}/\xi. \quad . \quad . \quad . \quad . \quad . \quad (12.0)$$

Let us now try to solve this equation, writing

$$\psi_2 = e^{-\eta^2/2}\phi(\xi).$$

It will then appear that η disappears from the equation on dividing by $e^{-\eta^2/2}$ and that ϕ satisfies the equation

$$\frac{d^2\phi}{d\xi^2} - \xi\frac{d\phi}{d\xi} - \phi = -\frac{L_1}{4\kappa^3 U}\cdot\frac{1}{\xi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.1)$$

of which the complete integral is easily found to be

$$\phi = -(L_1/4\kappa^3 U)\, e^{\xi^2/2}\int^{\xi}\log\xi e^{\xi^2/2}\,d\xi + Ce^{\xi^2/2}\int^{\xi} e^{-\xi^2/2}\,d\xi + De^{\xi^2/2},$$

and, since ϕ must necessarily be small when ξ is large and positive, the constants must be adjusted so that ϕ reduces to

$$\phi = (L_1/4\kappa^3 U)\, e^{\xi^2/2}\int_{\xi}^{\infty}\log\xi e^{-\xi^2/2}\,d\xi - Ce^{\xi^2/2}\int_{\xi}^{\infty} e^{-\xi^2/2}\,d\xi. \quad . \quad . \quad . \quad . \quad . \quad (12.2)$$

By integration by parts we readily obtain the asymptotic developments

$$\int_{\xi}^{\infty} e^{-\xi^2/2}\,d\xi = e^{-\xi^2/2}(\xi^{-1} - \xi^{-3} + \ldots),$$

$$\int_{\xi}^{\infty}\log\xi\, e^{-\xi^2/2}\,d\xi = e^{-\xi^2/2}\left\{\left(\frac{\log\xi}{\xi}\right) + \left(\frac{d}{\xi\,d\xi}\right)\left(\frac{\log\xi}{\xi}\right) + \left(\frac{d}{\xi\,d\xi}\right)^2\left(\frac{\log\xi}{\xi}\right) + \ldots\right\},$$

so that the more important terms in ϕ take the form

$$(L_1/4\kappa^3 U)(\log\xi/\xi) - C/\xi,$$

leading to

$$\psi_2 = (L_1/4\kappa^3 U)\, e^{-\eta^2/2}(\log\xi/\xi) - Ce^{-\eta^2/2}/\xi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.3)$$

which is identical with (11.6), C being suitably adjusted.

§ 13. *A Second Alternative Method of obtaining the* L_1*-Term.*

We may, however, arrive at this logarithmic term in yet another and more fundamental manner.

If we examine the constant L_1 (equations (5.43)), we notice that it contains the constant d_0 as a factor.

Accordingly, in $(u_0 + u_1)\dfrac{\partial\zeta_1}{\partial x} + (v_0 + v_1)\dfrac{\partial\zeta_1}{\partial y}$, this term can only arise from the d_0 terms in u_0, v_0. It must then come from the expression

$$R = d_0\left\{\frac{\sin\theta}{r}\frac{\partial\zeta_1}{\partial x} - \frac{\cos\theta}{r}\frac{\partial\zeta_1}{\partial y}\right\} = -\frac{d_0}{r^2}\frac{\partial\zeta_1}{\partial\theta}, \quad \ldots\ldots \quad (13.0)$$

whence, using (3.73),

$$R = -\frac{d_0}{r^2}\left\{\begin{array}{l} -\kappa r\sin\theta\,.\,e^{\kappa x}\sum\limits_{n=0}^{\infty}[a_n(S_{n+1}-S_n) + b_n(C_n - C_{n+1})] \\ + e^{\kappa x}\sum\limits_{n=0}^{\infty}[a_n\{(n+1)\,C_{n+1} - n\,C_n\} + b_n\{(n+1)\,S_{n+1} - n\,S_n\}] \end{array}\right\}. \quad (13.1)$$

Taking now the equation for ζ_2 (3.531) and writing

$$\zeta_2 = e^{\kappa x}\,.\,\zeta_2{}', \quad \ldots\ldots\ldots\ldots \quad (13.2)$$

we find that the corresponding part of $\zeta_2{}'$ satisfies the equation

$$\frac{\partial^2\zeta_2{}'}{\partial z^2} + \frac{1}{z}\frac{\partial\zeta_2{}'}{\partial z} + \frac{1}{z^2}\frac{\partial^2\zeta_2{}'}{\partial\theta^2} - \zeta_2{}'$$
$$= \frac{d_0}{\nu}\left[\begin{array}{l} \sin\theta\,\Sigma\,\{a_n(S_{n+1} - S_n)/z + b_n(C_n - C_{n+1})/z\} \\ -\,\Sigma\,\{a_n(\overline{n+1}\,C_{n+1} - n\,C_n)/z^2 + b_n(\overline{n+1}\,S_{n+1} - nS_n)/z^2\} \end{array}\right]. \quad (13.3)$$

The right-hand side of (13.3) may be simplified by using the difference equation for the K_n's, viz.,

$$2nK_n/z = K_{n+1} - K_{n-1}. \quad \ldots\ldots\ldots\ldots \quad (13.4)$$

It will be found to reduce to

$$-\frac{d_0}{2\nu z}\,\Sigma\,[a_n\{U_{n+1} + U_{n-1} - 2U_n\} + b_n\{V_{n+1} + V_{n-1} - 2V_n\}], \quad \ldots \quad (13.5)$$

where

$$\left.\begin{array}{l} U_n = K_{n+1}\cos n\theta + K_n\cos\overline{n+1}\,\theta \\ V_n = K_{n+1}\sin n\theta + K_n\sin\overline{n+1}\,\theta \end{array}\right\}. \quad \ldots\ldots\ldots \quad (13.6)$$

It will be observed that the typical equation to be solved is of the form

$$\frac{\partial^2\zeta_2{}'}{\partial z^2} + \frac{1}{z}\frac{\partial\zeta_2{}'}{\partial z} + \frac{1}{z^2}\frac{\partial^2\zeta_2{}'}{\partial\theta^2} - \zeta_2{}' = \frac{K_n}{z}\sin\{(n \pm 1)\,\theta + \varepsilon\}. \quad \ldots \quad (13.71)$$

To obtain the solution of this we consider first of all the more general equation

$$\frac{\partial^2 \zeta_2'}{\partial z^2} + \frac{1}{z}\frac{\partial \zeta_2'}{\partial z} + \frac{1}{z^2}\frac{\partial^2 \zeta_2'}{\partial \theta^2} - \zeta_2' = \frac{K_n}{z^p} \sin\{(n \pm p)\,\theta + \varepsilon\}, \quad . \;\; . \;\; . \;\; . \qquad (13.72)$$

where p is any constant, not necessarily integral.

It is easily verified, using the differential equation for the K_n's and the difference formulæ

$$K_n'(z) + K_{n\pm 1}(z) = \pm\, nK_n(z)/z \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \qquad (13.73)$$

that equation (13.72) has a particular integral

$$K_{n\pm 1} \sin\{(n \pm p)\,\theta + \varepsilon\}/2(p-1)z^{p-1}, \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \qquad (13.74)$$

upper and lower signs going together.

We note that when $p = 1$, which is precisely the case we require, this solution fails. If, however, we combine it with the complementary function

$$K_{n\pm p} \sin\{(n \pm p)\,\theta + \varepsilon\}/2\,(p-1),$$

then a solution is given by the expression

$$\underset{p\to 1}{\mathrm{L}} \sin\{(n \pm p)\,\theta + \varepsilon\}\left(\frac{K_{n\pm 1}}{z^{p-1}} - K_{n\pm p}\right)\Big/ 2\,(p-1),$$

i.e., by

$$-\tfrac{1}{2}\sin\{(n \pm 1)\,\theta + \varepsilon\}\,.\left\{K_{n\pm 1}\log z \pm \frac{\partial}{\partial n}K_{n\pm 1}\right\}. \quad . \;\; . \;\; . \;\; . \qquad (13.75)$$

Denoting now the function $\dfrac{\partial K_n}{\partial n}$ by H_n, we obtain the particular integral of (13.3),

$$\zeta_2' = \frac{d_0}{4\nu}\,\Sigma\,[a_n\{T_{n+1} + T_{n-1} - 2T_n\} + b_n\{W_{n+1} + W_{n-1} - 2W_n\}], \quad . \qquad (13.76)$$

where

$$\left.\begin{aligned} T_n &= (C_n + C_{n+1})\log z + H_{n+1}\cos\overline{n+1}\;\theta - H_n\cos n\theta \\ W_n &= (S_n + S_{n+1})\log z + H_{n+1}\sin\overline{n+1}\;\theta - H_n\sin n\theta \end{aligned}\right\} \quad . \;\; . \;\; . \qquad (13.77)$$

Thus the vorticity ζ_2 is obtained without approximation.

§ 14. *The Function* $H_n = \partial K_n/\partial n$.

The functions $\partial K_n/\partial n$ do not appear to have been studied, and it may be of some interest to develop here certain results which give information as to their magnitude and form.

If we differentiate the recurrence formula for the K_n's (13.4) with regard to n, we obtain

$$H_{n+1} - H_{n-1} = 2\,(K_n + nH_n)/z. \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \qquad (14.0)$$

Also K_n is known to be (and this is obvious from the asympototic development (5.0)) an even function of n. Thus H_n is an odd function of n and

$$H_{-n} = -H_n. \quad \ldots \quad (14.1)$$

Putting $n = 0$ into (14.0) and using (14.1), we find at once

$$H_1 = K_0/z, \quad \ldots \quad (14.21)$$

and putting $n = 0$ into (14.1)

$$H_0 = 0. \quad \ldots \quad (14.20)$$

The difference equation (14.0) then enables the successive H_n's to be readily calculated. Using the recurrence formula for the K's where necessary, we find

$$H_2 = 2K_1/z + 2K_0/z^2, \quad \ldots \quad (14.22)$$

$$H_3 = 3K_2/z + 6K_1/z^2 + 8K_0/z^3. \quad \ldots \quad (14.23)$$

This at once suggests a general form

$$H_n = A^n_{n-1} K_{n-1}/z + A^n_{n-2} K_{n-2}/z^2 + \ldots + A^n_0 K_0/z. \quad \ldots \quad (14.3)$$

In order to obtain the equations connecting successive A's, it will be convenient to use another recurrence formula, namely,

$$K_{n+1} = nK_n/z - \partial K_n/\partial z, \quad \ldots \quad (14.41)$$

leading to, on differentiation with regard to n,

$$H_{n+1} = K_n/z + nH_n/z - \partial H_n/\partial z. \quad \ldots \quad (14.42)$$

Substituting from (14.3) into (14.42) and expressing $\partial K_n/\partial z$ in terms of K_{n+1} and K_n/z by means of (14.41), we obtain

$$\begin{aligned} H_{n+1} = (A^n_{n-1} + 1) K_n/z &+ (2A^n_{n-1} + A^n_{n-2}) K_{n-1}/z^2 + \ldots \\ &+ (2pA^n_{n-p} + A^n_{n-p-1}) K_{n-p}/z^{p+1} + \ldots \\ &+ 2nA^n_0 K_0/z^{n+1}. \quad \ldots \quad (14.43) \end{aligned}$$

Hence

$$\left.\begin{aligned} A^{n+1}_n &= 1 + A^n_{n-1} \\ A^{n+1}_{n-1} &= 2A^n_{n-1} + A^n_{n-2} \\ &\ldots\ldots\ldots\ldots \\ A^{n+1}_{n-p} &= 2p\, A^n_{n-p} + A^n_{n-p-1} \\ &\ldots\ldots\ldots\ldots \\ A^{n+1}_0 &= 2n\, A^n_0 \end{aligned}\right\} \quad \ldots \quad (14.5)$$

From the last equation, since $A_0^1 = 1$,

$$A_0^{n+1} = 2^n \,.\, n\,! \quad . \quad . \quad . \quad . \quad (14.50)$$

The last equation but one of (14.5) then gives

$$A_1^{n+1} = 2\,(n-1)\,A_1^n + 2^{n-1}\,(n-1)\,!$$

or

$$\frac{A_1^{n+1}}{2^{n-1}\,(n-1)\,!} - \frac{A_1^n}{2^{n-2}\,(n-2)\,!} = 1, \quad \text{with } A_1^1 = 0,\ A_1^2 = 2.$$

This leads at once to

$$A_1^{n+1} = 2^{n-1}\,(n-1)\,!\,(n+1). \quad . \quad . \quad . \quad . \quad (14.51)$$

The last equation but two of (14.5) now gives

$$A_2^{n+1} = 2\,(n-2)\,A_2^n + 2^{n-2}\,(n-2)\,!\,n$$

or

$$\frac{A_2^{n+1}}{2^{n-2}\,(n-2)\,!} = \frac{A_2^n}{2^{n-3}\,(n-3)\,!} + n, \text{ with } A_2^3 = 3,$$

whence

$$A_2^{n+1} = 2^{n-2}\,(n-2)\,!\,\frac{n\,(n+1)}{1\,.\,2}. \quad . \quad . \quad . \quad . \quad (14.52)$$

This suggests the general formula

$$A_p^{n+1} = 2^{n-p}\,(n-p)\,!\,\frac{(n+1)\,(n)\,(n-1)\,\ldots\,(n-p+2)}{p\,!}. \quad . \quad . \quad (14.6)$$

Assuming this, the general equation in (14.5), which may be written

$$A_{p+1}^{n+1} = 2\,(n-p-1)\,A_{p+1}^n + A_p^n,$$

gives

$$\frac{A_{p+1}^{n+1}}{2^{n-p-1}\,(n-p-1)\,!} = \frac{A_{p+1}^n}{2^{n-p-2}\,(n-p-2)\,!} + \frac{n\,\ldots\,(n-p+1)}{p\,!}.$$

The last equation of this set refers to $p+1 = n-1$, or $n = p+2$, and gives

$$\frac{A_{p+1}^{p+3}}{2\,.\,1\,!} = A_{p+1}^{p+2} + \frac{(p+2)\,(p+1)\,\ldots\,3}{p\,!}.$$

But A_{p+1}^{p+2} is immediately obtained from the first of equations (14.5) and is equal to $p+2$.

By addition we find

$$\frac{A_{p+1}^{n+1}}{2^{n-p-1}\,(n-p-1)\,!}$$

$$= 1 + (p+1) + \frac{(p+2)\,(p+1)\,\ldots\,3}{p\,!} + \ldots + \frac{n\,(n-1)\,\ldots\,(n-p+1)}{p\,!}$$

$$= \frac{p\,! + (p+1)\,(p)\,(p-1)\,\ldots\,2 + (p+2)\,\ldots\,3 + \ldots + n\,(n-1)\,\ldots\,(n-p+1)}{p\,!},$$

the sum of which is well known to be

$$\frac{(n+1)\,n\,(n-1)\ldots(n-p+1)}{(p+1)\,!}.$$

This proves the formula (14.6) for $p+1$ if it holds for p, and we have proved it for the early values of p. As a check it is easily verified that it gives the correct value, $n+1$, when $p = n-1$.

We thus finally obtain the development

$$\mathrm{H}_n = {}_n\mathrm{C}_1\,\mathrm{K}_{n-1}/z + 2{}_n\mathrm{C}_2\mathrm{K}_{n-2}/z^2 + 2\,.\,4\,.\,{}_n\mathrm{C}_3\mathrm{K}_{n-3}/z^3 + \ldots + 2^{n-1}(n-1)\,!\,{}_n\mathrm{C}_n\mathrm{K}_0/z^n, \quad (14.7)$$

the ${}_n\mathrm{C}_p$'s having their usual meaning of the binomial coefficients.

Examination of this shows that when z is large the most important term in H_n is the leading one, so that we may write approximately

$$\mathrm{H}_n = n\mathrm{K}_{n-1}/z. \quad (14.8)$$

§ 15. *The Orders of the Terms in* ζ_2'.

Denoting for shortness by $\Delta^2 f(n)$ the expression

$$f(n+1) + f(n-1) - 2f(n),$$

let us examine the magnitudes of the terms in $\Delta^2\mathrm{T}_n$, $\Delta^2\mathrm{W}_n$. It is easy to verify, either from (14.7) or by differentiating with regard to n equation (5.0), that

$$\mathrm{H}_n = n\sqrt{\pi/2}\,e^{-z}z^{-3/2}\{1 + (4n^2-5)/8z\} \quad (15.0)$$

when z is large.

If we now bear in mind that, in the region where all these expressions are not exponentially small, θ is always of the order $1/\sqrt{z}$, and we develop in ascending powers of $1/z$, θ, grouping together terms of the same order, we obtain

$$\begin{aligned}\Delta^2(\mathrm{C}_n + \mathrm{C}_{n+1})\log z &\\ &= \log z\,[e^{-z}\sqrt{2\pi}\,z^{-\frac{1}{2}}(z^{-1} - \theta^2) + \text{terms of order } e^{-z}z^{-5/2}]\\ &= \text{an expression of order } e^{-z}z^{-3/2}\log z.\end{aligned}$$

$$\begin{aligned}\Delta^2\{\mathrm{H}_{n+1}\cos(n+1)\theta - \mathrm{H}_n\cos n\theta\} &\\ &= 3\sqrt{\pi/2}\,e^{-z}z^{-3/2}(z^{-1} - \theta^2) + \text{terms of order } e^{-z}z^{-7/2}\\ &= \text{an expression of order } e^{-z}\,.\,z^{-5/2}.\end{aligned}$$

The second term in $\Delta^2 T_n$ is therefore of order $1/z \log z$ of that of the leading term, and so may be neglected, when z in large, in comparison with this leading term.

Similarly we find

$$\Delta^2 (S_n + S_{n+1}) \log z$$
$$= \log z \left[\theta e^{-z} \sqrt{\pi/2}\, z^{-\frac{1}{2}} (3z^{-1} - \theta^2)(2n+1) + \text{terms of order } e^{-z} z^{-5/2}\right]$$
$$= \text{an expression of order } e^{-z} z^{-2} \log z.$$

$$\Delta^2 \{H_{n+1} \sin (n+1)\theta - H_n \sin n\theta\}$$
$$= \theta e^{-z} \sqrt{\pi/2}\, z^{-3/2} (6z^{-1} - 2\theta^2)(2n+1) + \text{terms of order } e^{-z} z^{-4}$$
$$= \text{an expression of order } e^{-z} z^{-3}.$$

Here again the second term may be neglected, in comparison with the leading term, so that we may take approximately, in (13.77)

$$T_n = (C_n + C_{n+1}) \log z \quad \ldots \ldots \ldots \quad (15.11)$$

$$W_n = (S_n + S_{n+1}) \log z. \quad \ldots \ldots \ldots \quad (15.12)$$

§ 16. *Approximate Expression for* ψ_2'.

The results of § 13 give us ζ_2, so far as the terms involving d_0 are concerned, without approximation.

To obtain the particular integral for ψ_2, we write $\psi_2 = e^{\kappa x} . \psi_2'$ in the equation

$$\nabla^2 \psi_2 = 2\zeta_2 = 2e^{\kappa x} . \zeta_2',$$

which now becomes

$$\nabla^2 \psi_2' + 2\kappa\, \partial \psi_2'/\partial x + \kappa^2 \psi_2' = 2\zeta_2',$$

and, on transforming to variables z and θ,

$$\left\{\frac{\partial^2}{\partial z^2} + \frac{1}{z}\frac{\partial}{\partial z} + \frac{1}{z^2}\frac{\partial^2}{\partial \theta^2} + 2\left(\cos\theta \frac{\partial}{\partial z} - \frac{\sin\theta}{z}\frac{\partial}{\partial \theta}\right) + 1\right\} \psi_2' = 2\zeta_2'/\kappa^2. \quad . \quad (16.0)$$

An exact solution of (16.0) would be difficult. We can, however, easily obtain an approximate solution by trial. If we substitute

$$\psi_2' = K_n . \log z . \sin (n\theta + \varepsilon)$$

and use the differential equation for K_n,

$$\frac{d^2 K_n}{dz^2} + \frac{1}{z}\frac{dK_n}{dz} - \left(\frac{n^2}{z^2} + 1\right) K_n = 0, \quad \ldots \ldots \quad (16.1)$$

and the recurrence formulæ

$$dK_n/dz - nK_n/z = -K_{n+1}, \quad \ldots \ldots \quad (16.2)$$

$$dK_n/dz + nK_n/z = +K_{n-1}, \quad \ldots \ldots \quad (16.3)$$

equation (16.0) will give

$$2\zeta_2'/\kappa^2 = -\Delta^2\{K_n \log z \sin(n\theta+\varepsilon)\} + (K_n/z)\{\sin[(n+1)\theta+\varepsilon] + \sin[(n-1)\theta+\varepsilon]\} - [(K_{n-1}+K_{n+1})\sin(n\theta+\varepsilon)]/z. \quad (16.4)$$

If in the above we write $\varepsilon = \pi/2$, the terms not involving $\log z$ on the right-hand side of (16.4) are

$$z^{-1}[K_n(\cos\overline{n+1}\,\theta + \cos\overline{n-1}\,\theta) - (K_{n-1}+K_{n+1})\cos n\theta]$$
$$= -e^{-z}\sqrt{\pi/2}\,z^{-3/2}\{z^{-1}+\theta^2\} + \text{smaller terms}$$
$$= \text{an expression of order } e^{-z}z^{-5/2},$$

which is of the same order as terms in the actual ζ_2' which we have seen to be negligible.

Similarly, if we write $\varepsilon = 0$ in (16.4), the terms free of $\log z$ are

$$z^{-1}[K_n(\sin\overline{n+1}\,\theta + \sin\overline{n-1}\,\theta) - (K_{n-1}+K_{n+1})\sin n\theta]$$
$$= -n\theta e^{-z}\sqrt{\pi/2}\,z^{-3/2}(z^{-1}+\theta^2) + \text{smaller terms}$$
$$= \text{an expression of order } e^{-z}z^{-3},$$

which is, again, negligible compared with the $\log z$ term, being of the same order as the terms $\Delta^2\{H_{n+1}\sin(n+1)\theta - H_n\sin n\theta\}$.

Accordingly, to this approximation, we have, if

$$\psi_2' = \{K_n\cos n\theta + K_{n+1}\cos(n+1)\theta\}\log z,$$
$$2\zeta_2'/\kappa^2 = -\Delta^2 T_n \text{ approximately; and if}$$
$$\psi_2' = \{K_n\sin n\theta + K_{n+1}\sin(n+1)\theta\}\log z$$
$$2\zeta_2'/\kappa^2 = -\Delta^2 W_n \text{ approximately.}$$

It follows that the approximate value of ψ_2' corresponding to equation (13.76) is given by

$$\psi_2' = -\frac{d_0}{2\nu\kappa^2}\log z\,\Sigma\,\{a_n(C_n+C_{n+1}) + b_n(S_n+S_{n+1})\}, \quad (16.5)$$

or, putting in the approximate values and retaining only the most important terms—remembering θ is of order $1/\sqrt{z}$,

$$\psi_2' = -\frac{d_0}{\nu\kappa^2}\Sigma\,a_n\,.\log z\,.\,e^{-z}\sqrt{\pi/2}\,.\,z^{-\frac{1}{2}}$$
$$= -\frac{\alpha_0 d_0}{\nu\kappa^2}\sqrt{\frac{\pi}{2}}\,.\,e^{-z}\,.\,z^{-\frac{1}{2}}\,.\log z.$$

Hence

$$\psi_2 = -[\alpha_0 d_0\sqrt{\pi/2})/\nu\kappa^2]\,e^{\kappa(x-r)}z^{-\frac{1}{2}}\log z. \quad (16.6)$$

Transforming to our previous co-ordinates ξ, η, this gives

$$\psi_2 = -(2\alpha_0 d_0 \sqrt{2\pi}/\nu\kappa^2)(e^{-\eta^2/2}\log\tfrac{1}{2}\xi)/\xi$$
$$= (\mathrm{L}_1/4\kappa^3\mathrm{U})(e^{-\eta^2/2}\log\tfrac{1}{2}\xi)/\xi, \qquad (16.7)$$

and this agrees with the values already found in (11.6) and (12.3).

It will be noticed that this last solution has now been obtained without having to adjust any discontinuities and has appeared as a natural solution of the differential equations.

§ 17. *Expressions for the Forces on the Cylinder.*

If we consider the rates of change of linear momentum of the fluid contained, at any moment, between the cylindrical obstacle and a large cylinder enclosing it, of which the cross-section in the x, y plane is a fixed contour Σ (ultimately taken as a circle of very large radius R), we obtain the following results:—

If X, Y are the forces applied by the fluid to the cylinder per unit length, and $\mathrm{X_B}$, $\mathrm{Y_B}$ are the total resolutes of the buoyancy of the cylinder and any cavities existing in the fluid inside Σ,

$$\mathrm{X_B} - \mathrm{X} = \dot{\mathrm{G}}_x + \rho\int_\Sigma \left\{ l\left(\frac{p}{\rho} + \Omega\right) + 2m\nu\zeta + u(lu + mv) \right\} ds, \qquad (17.01)$$

$$\mathrm{Y_B} - \mathrm{Y} = \dot{\mathrm{G}}_y + \rho\int_\Sigma \left\{ m\left(\frac{p}{\rho} + \Omega\right) - 2l\nu\zeta + v(lu + mv) \right\} ds, \qquad (17.02)$$

where G_x, G_y are the resolutes of linear momentum of the fluid inside Σ.

For the proof of these equations, the reader is referred to § 3 of "Forces."

In the present investigation the motion is supposed strictly steady, so that

$$\dot{\mathrm{G}}_x = \dot{\mathrm{G}}_y = 0.$$

Also

$$\frac{p}{\rho} + \Omega = \mathrm{Q} - \tfrac{1}{2}(u^2 + v)^2,$$

so that

$$\mathrm{X_B} - \mathrm{X} = \rho\int_\Sigma \{l\mathrm{Q} + 2m\nu\zeta + \tfrac{1}{2}l(u^2 - v^2) + muv\}\, ds, \qquad (17.11)$$

$$\mathrm{Y_B} - \mathrm{Y} = \rho\int_\Sigma \{m\mathrm{Q} - 2l\nu\zeta + luv - \tfrac{1}{2}m(u^2 - v^2)\}\, ds, \qquad (17.12)$$

and denoting by ds, dn the elements tangential and normal to the boundary of Σ,

$$lu + mv = u_n$$
$$lv - mu = u_s.$$

Thus (17.11) and (17.12) give

$$X_B - X = \rho \int (lQ + 2m\nu\zeta)\, ds + \tfrac{1}{2}\rho \int (uu_n - vu_s)\, ds \quad . \quad . \quad . \quad . \quad (17.21)$$

$$Y_B - Y = \rho \int (mQ - 2l\nu\zeta)\, ds + \tfrac{1}{2}\rho \int (uu_s + vu_n)\, ds. \quad . \quad . \quad . \quad . \quad (17.22)$$

The second integrals may be expressed in terms of disturbance velocities (u', v') as follows :—

$$u = U + u' \qquad v = v'$$

$$u_s = -mU + u_s' \qquad u_n = lU + u_n'$$

Thus

$$\int (uu_n - vu_s)\, ds = U \int u_n\, ds + \int (u'u_n - v'u_s)\, ds$$

$$= U \int u_n\, ds + U \int (lu' + mv')\, ds + \int (u'u_n' - v'u_s')\, ds$$

$$= U \left\{ \int_\Sigma u_n\, ds + \int_\Sigma u_n'\, ds \right\} + \int_\Sigma (u'u_n' - v'u_s')\, ds \quad . \quad . \quad . \quad . \quad (17.23)$$

But the condition that fluid cannot accumulate inside Σ gives

$$\int u_n\, ds = \int u_n'\, ds = 0.$$

Thus

$$\int (uu_n - vu_s)\, ds = \int (u'u_n' - v'u_s')\, ds.$$

Similarly,

$$\int (uu_s + vu_n)\, ds = U \int (u_s + u_s')\, ds + \int (u'u_s' + v'u_n')\, ds$$

and

$$\int u_s'\, ds = \int (-mU + u_s')\, ds = \int u_s\, ds = I,$$

where I is the circulation round Σ, so that

$$\int (uu_s + vu_n)\, ds = 2UI + \int_\Sigma (u'u_s' + v'u_n')\, ds. \quad . \quad . \quad . \quad . \quad (17.24)$$

Further, we may, in order to avoid solving for Q, express the leading integrals in (17.21), (17.22) exclusively in terms of ζ. For

$$\int_\Sigma lQ\, ds = \int_\Sigma Q\, dy = -\int_\Sigma y\, dQ$$

(owing to the one-valuedness of Q)

$$= -\int_\Sigma y \frac{\partial Q}{\partial s} \, . \, ds,$$

and, in like manner,

$$\int_\Sigma mQ\, ds = \int_\Sigma x \frac{\partial Q}{\partial s}\, ds.$$

But if we now refer to equations (3.06), (3.07) and take the x and y directions coincident with those of n and s, (3.07) gives

$$\frac{\partial Q}{\partial s} - 2\nu \frac{\partial \zeta}{\partial n} = -2u_n\zeta,$$

so that

$$\int_\Sigma lQ\, ds = -\int_\Sigma y\left(2\nu\frac{\partial \zeta}{\partial n} - 2u_n\zeta\right) ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.25)$$

$$\int_\Sigma mQ\, ds = +\int_\Sigma x\left(2\nu\frac{\partial \zeta}{\partial n} - 2u_n\zeta\right) ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.26)$$

and (17.21), (17.22) may now be written

$$X_B - X = \rho\int_\Sigma \left(2m\nu\zeta - 2\nu y\frac{\partial \zeta}{\partial n} + 2yu_n\zeta\right) ds + \tfrac{1}{2}\rho\int_\Sigma (u'u_n' - v'u_s')\, ds, \quad . \quad . \quad (17.31)$$

$$Y_B - Y = \rho\int_\Sigma \left(2\nu x\frac{\partial \zeta}{\partial n} - 2xu_n\zeta - 2l\nu\zeta\right) ds + \rho UI + \tfrac{1}{2}\rho\int_\Sigma (u'u_s' + v'u_n')\, ds. \quad (17.32)$$

Finally using

$$u_n = 2\kappa l\nu + u_n', \qquad u_s = -2\kappa m\nu + u_s'$$

we obtain

$$X - X_B = 2\rho\nu\int_\Sigma (y\,\partial\zeta/\partial n - 2\kappa y\, l\zeta - m\zeta)\, ds - 2\rho\int_\Sigma yu_n'\zeta\, ds$$
$$- \tfrac{1}{2}\rho\int_\Sigma (u'u_n' - v'u_s')\, ds \quad (17.41)$$

$$Y - Y_B = 2\rho\nu\int_\Sigma (2\kappa xl\zeta - x\,\partial\zeta/\partial n + l\zeta)\, ds + 2\rho\int_\Sigma xu_n'\zeta\, ds$$
$$- \tfrac{1}{2}\rho\int_\Sigma (u'u_s' + v'u_n')\, ds - \rho UI. \quad (17.42)$$

The equations (17.41), (17.42) give exact expressions for the forces on the cylinder, on the assumption of steady motion. The first terms on the right-hand side are of the first degree in the velocities, etc.; the second and third terms are of the second degree and were neglected in the original investigation ("Forces").

§ 18. *Expression for the Torque on the Cylinder.*

If we apply a similar calculation to the rate of change of the angular momentum about O of the fluid which occupies at time t the space W between Σ and the cylinder, we obtain

$$N' - N + N_1 + N_2 + \ldots + \iint_W (\rho y\,\partial\Omega/\partial x - \rho x\,\partial\Omega/\partial y)\, dx\, dy$$
$$= \dot{G} + \int_Z \rho\,(lu + mv)\,(xv - yu)\, ds, \quad (18.0)$$

where N is the moment about O of the forces applied by the fluid to the cylinder (per unit length of the latter), N′ is the moment of the tractions applied by the fluid outside

Σ to the fluid inside Σ, N_1, N_2 . . . are the moments of the forces exerted on this latter fluid by any hollow vortices. These are clearly zero, since the pressure inside such vortices may be treated as constant. Also, denoting by W_0 the *whole* space enclosed by Σ

$$\iint_W (\rho y\, \partial\Omega/\partial x - \rho x\, \partial\Omega/\partial y) dx\, dy$$

$$= \iint_{W_0} (\rho y\, \partial\Omega/\partial x - \rho x\, \partial\Omega/\partial y)\, dx\, dy$$

+ total moment about O of the buoyancies of cylinder and hollows

$$= \rho \int_\Sigma (ly - mx)\, \Omega\, ds + N_B, \text{ say,}$$

so that

$$N - N_B = N' - \dot{G} + \rho \int_\Sigma \{(ly - mx)\, \Omega - (lu + mv)\,(xv - yu)\}\, ds. \qquad (18.1)$$

Now

$$N' = \int_\Sigma \{x\,(l\,\widehat{xy} + m\,\widehat{yy}) - y\,(l\,\widehat{xx} + m\,\widehat{xy})\}\, ds$$

$$= \int_\Sigma (ly - mx)\, p\, ds + \mu \int_\Sigma \left\{x\left(l\frac{\partial u}{\partial y} + l\frac{\partial v}{\partial x} + 2m\frac{\partial v}{\partial y}\right)\right.$$

$$\left. - y\left(2l\frac{\partial u}{\partial x} + m\frac{\partial u}{\partial y} + m\frac{\partial v}{\partial x}\right)\right\} ds \qquad (18.2)$$

from equations (8.0).

The second integral in (18.2) may, on using the equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\zeta,$$

be written

$$2\int_\Sigma \left[\left\{-m\frac{\partial u}{\partial x} + l\frac{\partial u}{\partial y} + l\zeta\right\} x - \left\{m\frac{\partial v}{\partial x} - l\frac{\partial v}{\partial y} - m\zeta\right\} y\right] ds$$

$$= 2\int_\Sigma \left[x\frac{\partial u}{\partial s} + y\frac{\partial v}{\partial s} + (lx + my)\,\zeta\right] ds.$$

But

$$\int_\Sigma \left(x\frac{\partial u}{\partial s} + y\frac{\partial v}{\partial s}\right) ds = -\int_\Sigma \left(u\frac{dx}{ds} + v\frac{dy}{ds}\right) ds,$$

$$= -I,$$

I being, as before, the circulation round Σ.
Hence

$$N - N_B = -2\mu I - \dot{G} + \rho \int_\Sigma (ly - mx)\left(\Omega + \frac{p}{\rho}\right) ds$$

$$+ \rho \int_\Sigma \{2\nu\,(lx + my)\,\zeta - (lu + mv)\,(xv - yu)\}\, ds. \quad . \; . \; . \; . \; . \qquad (18.3)$$

This may be further simplified (i) by introducing the assumption of strictly steady motion, so that $\dot{G} = 0$; (ii) by taking the contour Σ to be a circle of radius R, centre the origin, so that $ly - mx = 0$, $lx + my = R$, $lu + mv = u_r$, $xv - yu = Ru_\theta$, $ds = R\,d\theta$.

We then have

$$(N - N_B)/\rho = -2\nu I + \int_0^{2\pi} (2\nu\zeta - u_r u_\theta)\, R^2\, d\theta. \quad . \quad . \quad . \quad . \quad . \quad (18.4)$$

Now

$$u_r u_\theta = -U^2 \cos\theta \sin\theta + U(u_\theta' \cos\theta - u_r' \sin\theta) + u_r' u_\theta'$$
$$= -U^2 \cos\theta \sin\theta + Uv' - 2u_r' U \sin\theta + u_r' u_\theta',$$

and, since $\int_0^{2\pi} \cos\theta \sin\theta\, d\theta = 0$,

$$(N - N_B)/\rho = -2\nu I + \int_0^{2\pi} (2\nu\zeta - Uv' + 2u_r' U \sin\theta - u_r' u_\theta')\, R^2\, d\theta$$
$$= -2\nu I + \int_0^{2\pi} \{2\nu(\zeta - \kappa v') + 2u_r' U \sin\theta - u_r' u_\theta'\}\, R^2\, d\theta. \quad . \quad . \quad (18.5)$$

If now we refer to the equations (3.6), (3.71), (3.72), which give the solution of the first order, it will be found that the velocities in polar co-ordinates are given by (see "Forces," equations (59) and (60), making the changes of notation indicated by the footnote to p. 5).

$$(u_r)_0 = -\alpha_0/\kappa^2 r + (c_1 \cos\theta + d_1 \sin\theta)/r^2 + 2(c_2 \cos 2\theta + d_2 \sin 2\theta)/r^3 + \ldots, \quad (18.61)$$

$$(u_\theta)_0 = -d_0/r + (c_1 \sin\theta - d_1 \cos\theta)/r^2 + 2(c_2 \sin 2\theta - d_2 \cos 2\theta)/r^3 + \ldots, \quad (18.62)$$

$$(u_r)_1 = (e^{\kappa x}/\kappa) \sum_{n=0}^{\infty} [a_n \{K_{n+1} \cos n\theta + K_n \cos(n+1)\theta\}$$
$$+ b_n \{K_{n+1} \sin n\theta + K_n \sin(n+1)\theta\}], \quad . \quad . \quad . \quad . \quad (18.63)$$

$$(u_\theta)_1 = (e^{\kappa x}/\kappa) \sum_{n=0}^{\infty} [a_n \{K_{n+1} \sin n\theta - K_n \sin(n+1)\theta\}$$
$$+ b_n \{K_n \cos(n+1)\theta - K_{n+1} \cos n\theta\}]. \quad . \quad . \quad . \quad . \quad . (18.64)$$

It appears therefore that, even if we consider only the terms $(u_r)_0$, $(u_\theta)_0$,

$$\int_0^{2\pi} (u_r)_0 (u_\theta)_0\, R^2\, d\theta = 2\pi\alpha_0 d_0/\kappa^2,$$

and does not vanish. Hence terms of the second degree can certainly not be neglected in the expression for the torque, even when we consider only the solution of the first order.

§ 19. *The Second Approximation to the Forces.*

In investigating the magnitudes of the various terms in the expressions for the forces, it will be desirable to tabulate the *orders* of the various terms.

To obtain these conveniently we use the approximate values of ψ_2 when r (and therefore ξ) is large, given by (9.9), as modified by (11.6), namely,

$$\psi_2 = (L_1/4\kappa^3U)\, e^{-\eta^2/2}\{\log(\xi/2)\}/\xi$$
$$+ (M_1/8\kappa^3U)\left[\left\{E(\eta) + \tfrac{1}{2}e^{-\eta^2/2}\int_0^\eta e^{-\eta^2/2}\,d\eta\right\}/\xi - \sqrt{\pi}\cos\tfrac{1}{2}\theta/4\sqrt{\kappa r}\right], \quad . \quad (19.0)$$

whence, from equations (4.33) and (4.34)

$$(u_r)_2 = -\partial\psi_2/r\partial\theta$$
$$= L_1\eta e^{-\eta^2/2}\log(\xi/2)/8\kappa^3 rU - \{(M_1\sqrt{\pi/\kappa})/64\kappa^3U\}\sin\tfrac{1}{2}\theta\,.\,r^{-3/2}$$
$$+ (M_1/32\kappa^3U)\left(e^{-\eta^2} + \eta e^{-\eta^2/2}\int_0^\eta e^{-\eta^2/2}\,d\eta\right)r^{-1}, \quad . \quad . \quad (19.1)$$

$$(u_\theta)_2 = \partial\psi_2/\partial r$$
$$= (L_1/8\kappa^3U)\, e^{-\eta^2/2}\{1 - (1+\eta^2)\log\tfrac{1}{2}\xi\}\,\xi^{-1}r^{-1}$$
$$+ (M\sqrt{\pi/\kappa}/64\kappa^3U)\cos\tfrac{1}{2}\theta\,.\,r^{-3/2}$$
$$- (M_1/32\kappa^3U)\left\{2E(\eta) + \eta e^{-\eta^2} + (1+\eta^2)\,e^{-\eta^2/2}\int_0^\eta e^{-\eta^2/2}\,d\eta\right\}r^{-1}\xi^{-1}, \quad . \quad . \quad (19.2)$$

$$\zeta_2 = (L_1/8\kappa^2U)(\eta^2 - 1)\,e^{-\eta^2/2}\log\tfrac{1}{2}\xi\,.\,r^{-1}\xi^{-1}$$
$$+ (M_1/32\kappa^2U)\left\{\eta e^{-\eta^2} + (\eta^2 - 1)\,e^{-\eta^2/2}\int_0^\eta e^{-\eta^2/2}\,d\eta\right\}\xi^{-1}r^{-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.3)$$

retaining only the most important terms.

The orders of magnitude and characteristics of evenness and oddness of the various terms in u_r, u_θ, ζ are given in the following table :—

TABLE. (19.4)

Expression.	Type of most important term.	
$(u_r)_0$, $(u_\theta)_0$	$1/r$	(19.41)
$(u_r)_1$	$e^{-\eta^2/2}\,.\,r^{-\frac{1}{2}}$ (even function of θ)	(19.42)
$(u_\theta)_1$	$e^{-\eta^2/2}\sin\theta r^{-\frac{1}{2}}$ (even function of θ)	(19.43)
$(u_r)_2$	$\sin\frac{1}{2}\theta\,.\,r^{-3/2}$, $\eta e^{-\eta^2/2}\log r/r$	(19.441), (19.442)
	$e^{-\eta^2/2}$ (even function of η)$/r$	(19.443)
$(u_\theta)_2$	$\cos\frac{1}{2}\theta\,.\,r^{-3/2}$, $e^{-\eta^2/2}$ (even function of η) $\log r/r^{3/2}$	(19.451), (19.452)
	$e^{-\eta^2/2}$ (odd function of η)$/r^{3/2}$	(19.453)
ζ_1	$e^{-\eta^2/2}\sin\theta\, r^{-\frac{1}{2}}$ (even function of θ)	(19.46)
ζ_2	$(1-\eta^2)\,e^{-\eta^2/2}\log r\,.\,r^{-3/2}$, $e^{-\eta^2/2}$ (odd function of η) $r^{-3/2}$	(19.471), (19.472)

In order to make the discussion more intelligible, we shall denote the parts of the integrals on the right-hand sides of (17.41), (17.42), (18.5), which involve the velocities and vorticity to the *first degree* by X_0, Y_0, N_0 ; X_1, Y_1, N_1, and X_2, Y_2, N_2 according as we substitute in them the values with suffixes 0, 1, 2 respectively, we shall denote the integrals of the second degree by X_{pq}, Y_{pq}, N_{pq} (p, $q = 0$, 1, 2) the first suffix corresponding to the substitution into the *first* factor of each term, the second suffix to the substitution into the *second* factor of each term.

In the case of the forces, there are two integrals of second degree, corresponding to the second and third integrals on the right-hand side of (17.41), (17.42). These will be denoted by one and two accents respectively, so that

$$X_{01}' = -2\rho \int_{\Sigma} y (u_r)_0 \zeta_1 \, ds,$$

$$Y_{22}'' = -\tfrac{1}{2} \rho \int_{\Sigma} \{u_2 (u_\theta)_2 + v_2 (u_r)_2\} \, ds.$$

With the above notation we proceed to examine the forces. Since

$$\zeta_0 = 0, \qquad X_0 = Y_0 = 0,$$

X_1 and Y_1 have been calculated in the previous paper ("Forces") and the results are known to be finite.

For X_2 and Y_2, we note that

$$\frac{\partial \zeta_2}{\partial r} = \frac{1}{2r}\left(\xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta}\right) \zeta_2,$$

so that the most important term in $\partial \zeta_2 / \partial r$ is of type

$$f(\eta^2)\, e^{-\eta^2/2} \log r \,.\, r^{-5/2},$$

where f is a finite polynomial. Hence

$$\int y \frac{\partial \zeta_2}{\partial r} r \, d\theta = \int r^2 \,.\, \frac{\partial \zeta_2}{\partial r} \,.\sin \theta \, d\theta = (r^{-\frac{1}{2}} \log r) \int f(\eta^2)\, e^{-\eta^2/2} \sin \theta \, d\theta,$$

which vanishes unconditionally when r is large.

Similarly $\int x \frac{\partial \zeta_2}{\partial r} \, ds$ vanishes when r is large.

For a similar reason $\int m\zeta \, ds$ and $\int l\zeta \, ds$ both vanish unconditionally when r is large.

Considering next $\int yl\zeta \, ds$, which occurs in X_2, it appears that this is of form

$$r^{\frac{1}{2}} \log r \int \cos \theta \sin \theta \,.\, (1 - \eta^2)\, e^{-\eta^2/2} \, d\theta.$$

Now the only important part of this integral is near $\theta = 0$, where $\cos \theta \sin \theta$ is approximately equal to $2\eta/\xi$ and $d\theta$ is $2\, d\eta/\xi$. It follows that the integral approximates to

$$4 r^{\frac{1}{2}} \xi^{-2} \log r \int \eta\, (1 - \eta^2)\, e^{-\eta^2/2} \, d\eta.$$

The integrand being *odd*, the integral would vanish in any case; apart from this, however, it vanishes in virtue of its order, since $\xi^2 = 4\kappa r$ here and the factor outside is of magnitude $(\log r)\, r^{-\frac{1}{2}}$.

The case of $\int xl\zeta\, ds$ requires special attention. This reduces approximately to

$$2r^{\frac{1}{2}}\xi^{-1}\log r \int_{-\infty}^{\infty} (1-\eta^2)\, e^{-\eta^2/2}\, d\eta$$

and the factor outside is of order $\log r$. This does not vanish *unconditionally*, that is, in virtue of its order alone, but since $\int^{\eta} (1-\eta^2)\, e^{-\eta^2/2}\, d\eta = \eta\, e^{-\eta^2/2}$, it is easily seen that $\int_{-\infty}^{\infty} (1-\eta^2)\, e^{-\eta^2/2}\, d\eta = 0$. The integral thus vanishes *conditionally*; and it follows that, since any inverse power of r will ultimately, when multiplied by $\log r$, lead to a vanishing quantity, the next terms in ζ_2 must, when substituted into the integral, make it vanish unconditionally.

Thus X_2 and Y_2 both vanish.

We now pass on to the consideration of the terms of second degree with one accent. Clearly we need only examine

$$X_{01}',\ X_{02}',\ X_{11}',\ X_{12}',\ X_{21}',\ X_{22}'$$

and the corresponding Y_{pq}'s, since those with second suffix 0 vanish identically.

Now

$$\int yu'_n\zeta\, ds = r^2 \int \sin\theta\, .\, u_r'\, .\, \zeta\, .\, d\theta,$$

and

$$\int xu'_n\zeta\, ds = r^2 \int \cos\theta\, .\, u_r'\, .\, \zeta\, .\, d\theta.$$

Owing to the presence of ζ, a negative exponential factor occurs in every case, so that we can write, as before, $d\theta = 2d\eta/\xi$, $\sin\theta = 2\eta/\xi$, $\cos\theta = 1$, and the integrals approximate to

$$\frac{r}{\kappa}\int \eta u_r'\zeta\, d\eta,$$

$$\frac{r^{3/2}}{2\sqrt{\kappa}}\int u_r'\zeta\, d\eta.$$

The values in the table immediately show that the expressions of type (0, 2), (1, 2), (2, 2) vanish unconditionally. In the case of type (0, 1) we have to remember that $\sin\theta$ is here $2\eta/\xi$, so that $(u_r)_0\, \zeta_1$ is of order $1/r^2$. Hence X_{01}', Y_{01}' both vanish unconditionally.

For the type (1, 1) $(u_r)_1\, \zeta_1$ is of order $r^{-3/2}$ (odd function of η). Thus X_{11}' vanishes unconditionally, but Y_{11}' vanishes conditionally—being, however, unconditionally finite, which guarantees that the next terms in the approximation must vanish unconditionally. Finally, for type (2, 1) $(u_r)_2\ \zeta_1$ is of order $\log r \,.\, r^{-2}$ and the integrals both vanish unconditionally.

It therefore appears that this second approximation to the solution introduces no modification to the forces on the cylinder calculated in "Forces," and the simple expressions for the Drag and Lift, obtained in the previous paper, hold good unmodified.

§ 20. *The Torque as given by the First Approximation.*

We proceed to calculate the torque in the same way. The integral of the first degree (from (18.5)) is

$$r^2 \int_0^{2\pi} [2\nu\,(\zeta - \kappa v') + 2u_r' \mathrm{U} \sin\theta]\, d\theta, \quad \text{(20.0)}$$

and the integral of the second degree is

$$- r^2 \int_0^{2\pi} u_r' u_\theta'\, d\theta. \quad \text{(20.1)}$$

Thus

$$\mathrm{N}_0 = r^2 \int_0^{2\pi} [-\,2\nu\kappa v_0 + 2\,(u_r)_0\, \mathrm{U} \sin\theta]\, d\theta,$$

where, from (3.6), (18.61) and (18.62)

$$v_0 = -\,\alpha_0 \sin\theta/\kappa^2 r - d_0 \cos\theta/r + (c_1 \sin 2\theta - d_1 \cos 2\theta)/r^2,$$

$$(u_r)_0 = -\,\alpha_0/\kappa^2 r + (c_1 \cos\theta + d_1 \sin\theta)/r^2,$$

$$(u_\theta)_0 = -\,d_0/r + (c_1 \sin\theta - d_1 \cos\theta)/r^2,$$

terms in r^{-3}, etc., being immaterial.

On substitution, we find

$$\mathrm{N}_0 = 2\pi \mathrm{U} d_1 \quad \text{(20.21)}$$

$$\mathrm{N}_{00} = -\,2\pi\alpha_0\, d_0/\kappa^2. \quad \text{(20.22)}$$

The term N_0 leads to the ordinary formula for the torque in the case of irrotational motion of a perfect fluid

Since $\zeta_1 - \kappa v_1 = 0$, N_1 reduces to

$$r^2 \int_0^{2\pi} 2(u_r)_1\, \mathrm{U} \sin\theta\, d\theta = r^2 \int_{-\pi}^{+\pi} 2(u_r)_1\, \mathrm{U} \sin\theta\, d\theta.$$

If we now refer to (18.63), we see that the above integral contains the usual negative exponential factor. If we then develop (18.63) in descending powers of r, bearing in mind that $\sin\theta\, d\theta = 4\eta\, d\eta/\xi^2$ approximately, it will appear that we can replace the K_n's in

(18.63) by the first term of their asymptotic development, and, further, that we can write $\cos n\theta = 1$ and $\sin n\theta = n\theta$ to the same approximation. We then have

$$(u_r)_1 = (\sqrt{2\pi}/\kappa)\, e^{-\eta^2/2}\, \xi^{-1}\, (2\alpha_0 + \theta\beta),$$

β having the value $(\Sigma\,(2n+1)\,b_n)$ given by (5.23).

Thus

$$N_1 = (8r^2\sqrt{2\pi}/\kappa\xi^3) \int_{-\infty}^{\infty} U e^{-\eta^2/2} \eta\, d\eta\, (2\alpha_0 + \theta\beta).$$

The term involving α_0 vanishes conditionally, the integrand being odd, and the term in β gives

$$\begin{aligned} N_1 &= (16r^2\sqrt{2\pi}\, U\beta/\kappa\xi^4) \int_{-\infty}^{\infty} e^{-\eta^2/2}\eta^2 d\eta \\ &= 32\pi\, U\beta r^2/16\kappa^3 r^2 \\ &= 2\pi U\beta/\kappa^3. \qquad (20.23) \end{aligned}$$

Before proceeding to investigate the terms with suffix 2, let us examine the terms N_{01}, N_{10}, N_{11}. From these we will obtain what the torque on the cylinder would have been if the Oseen approximation (what we have called the first order terms) had been strictly correct.

Since we are here necessarily introducing exponential factors, we may write

$$N_{pq} = -\frac{\xi^3}{8\kappa^2} \int_{-\infty}^{\infty} (u_r)_p\, (u_\theta)_q\, d\eta.$$

Thus

$$N_{01} = +\,\xi^3 \,.\, (\alpha_0/8\kappa^4)\, r^{-1} \int_{-\infty}^{\infty} (u_\theta)_1\, d\eta.$$

Now, if we treat $(u_\theta)_1$ as we did $(u_r)_1$, it appears that

$$(u_\theta)_1 = -(\alpha_0\sqrt{2\pi}/\kappa)\, e^{-\eta^2/2}\, \theta/\xi = -(2\alpha_0\sqrt{2\pi}/\kappa)\, e^{-\eta^2/2}\, \eta/\xi^2,$$

and, on substitution, it is found that N_{01} vanishes unconditionally when r, that is ξ, tends to infinity.

Again,

$$\begin{aligned} N_{10} &= +\,(d_0\sqrt{2\pi}/8\kappa^3)\,(\xi^2/r) \int_{-\infty}^{\infty} e^{-\eta^2/2}\,(2\alpha_0 + \theta\beta)\, d\eta \\ &= 2\pi\alpha_0 d_0/\kappa^2, \qquad (20.24) \end{aligned}$$

so that $N_{00} + N_{10} = 0$.

Finally,

$$\begin{aligned} N_{11} &= \alpha_0\pi/2\kappa^4 \int_{-\infty}^{\infty} e^{-\eta^2}\,(2\alpha_0 + \theta\beta)\,\eta\, d\eta \qquad (20.25) \\ &= 0 \text{ conditionally,} \end{aligned}$$

the only integral of finite order vanishing on account of oddness.

We thus have finally, for the resistance torque on the cylinder, as given by the first order solution

$$(\mathrm{N}-\mathrm{N_B})/\rho = -2\nu\mathrm{I} + 2\pi\mathrm{U}d_1 + 2\pi\mathrm{U}\beta/\kappa^3. \quad \ldots \quad (20.3)$$

The last term in (20.3) has an interesting physical interpretation.

It can be shown that the rotational terms of the first order give no circulation round Σ.

For such a circulation would be given by

$$\mathrm{J} = \int_0^{2\pi} r(u_\theta)_1\,d\theta = -(\alpha_0\sqrt{2\pi}/\kappa^2\xi)\int_{-\infty}^{\infty} e^{-\eta^2/2}\,\eta\,d\eta$$
$$= 0. \quad \ldots \quad (20.4)$$

If, however, we take the development of $(u_\theta)_1$ one step farther, we find that

$$(u_\theta)_1 = -(2\sqrt{2\pi}/\kappa)\,e^{-\eta^2/2}\{\alpha_0\eta/\xi^2 + \beta(1+\eta^2)/\xi^3\}$$

and

$$r\mathrm{J} = \int_0^{2\pi} r^2(u_\theta)_1\,d\theta = -(\sqrt{2\pi}/4\kappa^3)\int_{-\infty}^{\infty}(\alpha_0\xi\eta e^{-\eta^2/2} + \beta(1+\eta^2)\,e^{-\eta^2/2})\,d\eta$$
$$= -\pi\beta/\kappa^3, \quad \ldots \quad (20.5)$$

so that $r\mathrm{J}$ tends to a finite limit, where J is the circulation across the tail.

This gives for the torque the expression

$$(\mathrm{N}-\mathrm{N_B})/\rho = -2\nu\mathrm{I} + 2\pi\mathrm{U}d_1 - 2\mathrm{U}(r\mathrm{J}). \quad \ldots \quad (20.6)$$

The constant d_1, however, will depend on the attitude and shape of the obstacle, and is not necessarily the same as the corresponding constant in the irrotational motion of a perfect fluid past the same obstacle.

§ 21. *The Second Approximation for the Torque.*

We now examine the terms in N involving suffix 2, and it will be convenient to begin with the terms of second degree.

Examination of the table (19.4) shows that:—

$(u_r)_0\,(u_\theta)_2$ is of order $r^{-5/2}$ or $\log r\,.\,r^{-5/2}$, so that when r is large $\mathrm{N}_{02}=0$.

$(u_r)_1\,(u_\theta)_2$ is of order $e^{-\eta^2/2}\,.\,r^{-2}$ or $e^{-\eta^2/2}\log r\,.\,r^{-2}$; in this case, owing to the presence of the exponential factor, we can restrict the range in θ, and $d\theta = 2d\eta/\xi = d\eta/\sqrt{\kappa r}$, which gives an additional factor $r^{-1/2}$, so that N_{12} vanishes unconditionally.

$(u_r)_2\,(u_\theta)_2$ is of order $e^{-\eta^2}(\log r)^2\,r^{-5/2}$, or $e^{-\eta^2/2}(\log r)\,r^{-5/2}$ or smaller orders. It is at once obvious that N_{22} vanishes unconditionally.

$(u_r)_2\,(u_\theta)_0$ is of order $\eta e^{-\eta^2/2}\log r/r^2$, or $e^{-\eta^2/2}/r^2$, or $\sin\frac{1}{2}\theta r^{-5/2}$ and, since $d\theta = d\eta/\sqrt{\kappa r}$ when the exponential factor is present, N_{20} vanishes unconditionally.

$(u_r)_2\,(u_\theta)_1$ has its most important terms of type

$$e^{-\eta^2/2}\sin\theta\sin\tfrac{1}{2}\theta\,.\,r^{-2},\quad \eta e^{-\eta^2}\log r\,.\sin\theta\,.\,r^{-3/2},\quad e^{-\eta^2}\sin\theta\,r^{-3/2}.$$

In virtue of the fact that θ is of order $r^{-\frac{1}{2}}$ (since the negative exponential factor is throughout present) the first of these leads to an integral vanishing unconditionally. The second does likewise, writing $\sin\theta = \eta/\sqrt{\kappa r}$, $d\theta = d\eta/\sqrt{\kappa r}$, and so does the third. Thus N_{22} vanishes unconditionally.

Accordingly, the whole of the terms of the second degree really contribute (having regard to the results of § 20) nothing to the torque on the cylinder.

There remains the term N_2 of the first degree. This is given by

$$N_2 = r^2 \int_0^{2\pi} [2\nu\,(\zeta_2 - \kappa v_2) + 2\,(u_r)_2\,U \sin\theta]\,d\theta.$$

If we consider first the integral

$$2\nu \int_\Sigma (\zeta_2 - \kappa v_2)\,r^2\,d\theta$$

and refer to equation (9.91), we find, from the last term in (9.91),

$$-\,(M_1\sqrt{\pi}/64\kappa^{7/2})\,r^{1/2}\int_0^{2\pi}\cos\tfrac{3}{2}\,\theta d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21.0)$$

This happens to vanish (conditionally); and, as it refers to a part of the solution which is exact, we need not enquire about the next term in the approximation. In any case, if a term of this type occurred when terms of lower order in ξ are included, it is clear that such a term would be of order at least one lower in ξ than the one here retained, so that it would lead to an integral of order zero in r, which would at most be finite.

The other terms in (9.91) lead to

$$-\,(L_1/2\kappa^2)\int_\Sigma r^2\xi^{-3}e^{-\eta^2/2}\,d\theta = -\,(L_1/16\kappa^4)\int_{-\infty}^{\infty} e^{-\eta^2/2}\,d\eta = -\,\sqrt{2\pi}\,L_1/16\kappa^4 \quad . \quad (21.1)$$

and

$$(M_1/4\,\kappa^2)\int_\Sigma r^2\xi^{-3}E\,(\eta)\,d\theta = (M_1/32\kappa^4)\int_{-\infty}^{\infty} E\,(\eta)\,d\eta = 0. \quad . \quad . \quad . \quad . \quad . \quad (21.2)$$

Both these are of order zero, so that any succeeding terms would be of order $r^{-1/2}$ at most, and so negligible.

Coming now to the integral

$$\int_\Sigma 2U\,r^2\,(u_r)_2\,\sin\theta d\theta,$$

we take our value of $(u_r)_2$ from (19.1).

The middle term in the right-hand side of (19.1) leads to

$$-\,(M_1\,\sqrt{\pi}/32\kappa^{7/2})\,r^{1/2}\int_0^{2\pi}\sin\tfrac{1}{2}\theta\,\sin\theta d\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21.3)$$

and this vanishes conditionally, the same remarks applying to it as to (21.0) above.

The last term in the right-hand side of (19.1) leads to

$$(M_1/16\kappa^3)\int_\Sigma r\,\sin\,\theta\,d\theta\left(e^{-\eta^2} + \eta e^{-\eta^2/2}\int_0^{\eta} e^{-\eta^2/2}\,d\eta\right). \quad . \quad . \quad . \quad . \quad . \quad (21.4)$$

This is of order zero, so that the next approximation is negligible, and it vanishes conditionally, owing to the oddness in η of the integrand.

Finally, we have, from the L_1-term in (19.1),

$$(L_1/4\kappa^3)\int_\Sigma r\eta\, e^{-\eta^2/2}\log \tfrac{1}{2}\xi \,.\, \sin\theta\, d\theta$$

$$= (L_1/4\kappa^4)\left(\int_{-\infty}^{\infty}\eta^2\, e^{-\eta^2/2}\, d\eta\right).\,(\log \tfrac{1}{2}\xi)$$

$$= (L_1\sqrt{2\pi}/4\kappa^4)\log \tfrac{1}{2}\xi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21.5)$$

This not only does not vanish, but becomes logarithmically infinite when ξ (*i.e.*, r) becomes infinite.

It will be clear from the discussion which has been given that none of the other terms retained can possibly cancel this, as this is the only term which does not give a finite result. Moreover, bearing in mind that the next terms (which have not been retained) are of order at least one higher in $1/\xi$, *i.e.*, $1/r^{\frac{1}{2}}$, it also appears that the terms neglected in the course of the present second order approximation could not possibly give rise to terms in the torque of such magnitude as to cancel (21.5).

It would therefore appear that this method of approximation necessarily leads to an impossible result.

§ 22. *Discussion of Reasons for Failure.*

The possible explanations of the appearance of the physically inadmissible result of the last section appear to reduce to four :—

(i) That if the method of approximation adopted were continued so as to include third and higher order terms, a term would arise which would cancel (21.5).

(ii) That the particular integrals selected for ψ_2, ζ_2 in §§ 6, 7 are unsuitable and that other particular integrals would lead to an admissible result.

(iii) That *steady* motion of the type postulated is in this case impossible theoretically (apart from the question of its stability).

(iv) That there is a fundamental vice in the method of approximation adopted, and that $(u_0 + u_1,\ v_0 + v_1)$ do not supply a sound first approximation for further developments.

A little consideration of dimensions will show that (i) must be dismissed. For, apart from the improbability that such third order developments could provide a term of sufficiently high order in r—and it must be remembered that the objectionable term is really of so low an order of infinity that almost any reduction of order will cause it to vanish—there is the further consideration that terms of the third order will involve the constants a_n, b_n, c_n, d to the third degree. Any *identical* linear relation between such terms and a term of the second degree in the same constants, such as the one in question, is clearly impossible. Yet, since these constants are all arbitrary and independent, such a relation, if it existed, would have to be *identical*. Similar reasoning applies to fourth and higher order solutions.

It also appears that (ii) can hardly be accepted as an explanation. In the first place, we note that this particular integral has been arrived at in three entirely distinct ways, all of which have led to the same result. This, in itself, would not be mathematically conclusive. But if the logarithmic term which causes the trouble could be removed by selecting another particular integral, it would follow, by taking the difference of the two particular integrals, that a complementary function would exist, satisfying all the conditions of continuity and containing, unbalanced, the logarithmic term in question. Such a complementary function would necessarily lead to a solution of first order. Now the solutions of first order, which satisfy the necessary conditions of continuity, have been carefully enumerated in the paper referred to as "Forces," and such a solution involving $\log z$ is not among them. Solutions of the equation

$$\nabla^2\zeta - 2\kappa \frac{\partial\zeta}{\partial x} = 0$$

involving $\log z$ do exist, but they all involve discontinuities or many-valuedness in θ.

Coming now to (iii) the reasons against this have already been given in § 2. It would amount to a difficulty analogous to that of STOKES when the inertia-terms in two-dimensional motion are neglected. A difficulty of this class appears to be inherent in the approximate type of equations selected, and one can hardly admit that it is essential to the physics of the problem. STOKES' difficulty disappears if we replace his equations by those of OSEEN, and there seems no reason to doubt that the use of more exact equations would remove also the present one. Looked at from this point of view, (iii) is largely identical with (iv).

We are thus almost inevitably driven to the conclusion (iv), that the method of approximation is unsound. Now examination of the expressions (19.1), (19.2) will show that the most important terms in our second approximation for the velocities are, in fact, of smaller order than the most important terms in the first approximation.

Thus away from the tail the most important terms of second order are of magnitude $r^{-3/2}$, those of the first order of magnitude r^{-1}; in the tail the most important terms of second order are of magnitude $(\log r/r)$, those of the first order of magnitude $r^{-\frac{1}{2}}$. There is thus, so far as we have gone, no apparent failure of convergency.

The approximation appears to fail in virtue of its having been applied to the terms in α_0, d_0 in the first order solution. We cannot get over the difficulty by making α_0, $d_0 = 0$, for $\alpha_0 = 0$ would cause the drag to vanish, and $d_0 = 0$ would cause the lift to vanish (see "Forces").

It seems probable that our first order solution based upon equations (3.42), (3.52) (which we may refer to for brevity as the "Oseen" solution) fails in some way to take account of effects which should be included at the very start, and that the true "first order" solution should be based upon a closer approximation to the actual motion at a great distance.

An analogous difficulty which occurs in the Lunar Theory has been referred to in § 2.

§ 23. *Proposed New Fundamental Equations.*

Since the difficulty has arisen over the terms in α_0, d_0, which give the most important irrotational terms when r is large, the following modified line of attack suggests itself.

Instead of writing

$$u = \mathrm{U} + u'$$
$$v = v'$$

let us write

$$\left.\begin{aligned} u &= (\mathrm{U} + u_0) + u' \\ v &= v_0 + v' \end{aligned}\right\}, \qquad (23.0)$$

where u_0, v_0 are the velocities due to α_0, d_0. If the Oseen solution gives any approximation at all away from the tail, then u', v' should now be, in general, of very much smaller magnitude than before.

If we assume steady motion, we find as before

$$\frac{\partial \mathrm{Q}}{\partial x} + 2\nu \frac{\partial \zeta}{\partial y} = 2v\zeta,$$

$$\frac{\partial \mathrm{Q}}{\partial y} - 2\nu \frac{\partial \zeta}{\partial x} = -2u\zeta,$$

and

$$\nu \nabla^2 \zeta = u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y}.$$

ζ involves only u', v', and if we *now* neglect terms of second order in u', v', we have, for the fundamental equation for the vorticity,

$$\nu \nabla^2 \zeta = (\mathrm{U} + u_0) \frac{\partial \zeta}{\partial x} + v_0 \frac{\partial \zeta}{\partial y}, \qquad (23.1)$$

or, writing $\mathrm{U} = 2\nu\kappa$ as before,

$$\begin{aligned} \nu (\nabla^2 \zeta - 2\kappa\, \partial\zeta/\partial x) &= u_0\, \partial\zeta/\partial x + v_0\, \partial\zeta/\partial y \\ &= (u_r)_0\, \partial\zeta/\partial r + (u_\theta)_0\, \partial\zeta/r d\theta \\ &= -(\alpha_0/\kappa^2)\, \partial\zeta/r\partial r - d_0\, \partial\zeta/r^2\partial\theta, \end{aligned}$$

or, writing for shortness $\alpha_0/\kappa^2\nu = g_1$, $d_0/\nu = g_2$, we have

$$\nabla^2 \zeta - 2\kappa \frac{\partial \zeta}{\partial x} + \frac{g_1}{r}\frac{\partial \zeta}{\partial r} + \frac{g_2}{r^2}\frac{\partial \zeta}{\partial \theta} = 0. \qquad (23.2)$$

The differential equation (23.2) does not seem to have been studied. It will be necessary to investigate its solutions before applications to the problem of viscous flow can be made.

Such an investigation will not be attempted here, but it may not be without interest to note the following indication of the type of solution which may be expected:—

(23.2) may be written

$$\frac{\partial^2\zeta}{\partial r^2} + \frac{(1+g_1)}{r}\frac{\partial\zeta}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2\zeta}{\partial\theta^2} + g_2\frac{\partial\zeta}{\partial\theta}\right) = 2\kappa\frac{\partial\zeta}{\partial x}, \quad \ldots \quad (23.3)$$

and writing $x = hx'$, $y = hy'$, $r = hr'$, this becomes

$$\frac{\partial^2\zeta}{\partial r'^2} + \frac{(1+g_1)}{r'}\frac{\partial\zeta}{\partial r'} + \frac{1}{r'^2}\left(\frac{\partial^2\zeta}{\partial\theta^2} + g_2\frac{\partial\zeta}{\partial\theta}\right) = 2\kappa h\frac{\partial\zeta}{\partial x'}, \quad \ldots \quad (23.4)$$

which is of the same form as (23.3), with κh written for κ. Since h may have any value, we therefore lose no generality by assuming κ in (23.3) to be very small.

If we now assume a solution in ascending powers of κ,

$$\zeta = Z_0 + \kappa Z_1 + \kappa^2 Z_2 + \kappa^3 Z_3 + \ldots \quad \ldots \quad (23.5)$$

we obtain the series of equations

$$\frac{\partial^2 Z_0}{\partial r^2} + \frac{1+g_1}{r}\frac{\partial Z_0}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 Z_0}{\partial\theta^2} + g_2\frac{\partial Z_0}{\partial\theta}\right) = 0, \quad \ldots \quad (23.50)$$

$$\frac{\partial^2 Z_1}{\partial r^2} + \frac{1+g_1}{r}\frac{\partial Z_1}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 Z_1}{\partial\theta^2} + g_2\frac{\partial Z_1}{\partial\theta}\right) = 2\frac{\partial Z_0}{\partial x}, \quad \ldots \quad (23.51)$$

etc., from which Z_0, Z_1, Z_2, etc., may be successively determined.

If in the first equation we write

$$Z_0 = R_0 e^{in\theta}, \quad \ldots \quad (23.6)$$

where R_0 is a function of r only, we obtain

$$d^2R_0/dr^2 + (1+g_1)\,dR_0/rdr + (-n^2 + ing_2)\,R_0/r^2 = 0, \quad \ldots \quad (23.7)$$

of which the solution is known to be of the type

$$R_0 = r^p, \quad \ldots \quad (23.71)$$

and we have

$$p^2 + g_1 p - n^2 + ing_2 = 0, \quad \ldots \quad (23.72)$$

leading to

$$p = -\tfrac{1}{2}g_1 \pm \rho_n e^{-i\alpha_n}, \quad \ldots \quad (23.73)$$

where

$$\rho_n^2 \cos 2\alpha_n = n^2 + g_1^2/4 \quad \ldots \quad (23.74)$$

$$\rho_n^2 \sin 2\alpha_n = ng_2. \quad \ldots \quad (23.75)$$

We thus obtain, as a typical solution for Z_0,

$$Z_0 = r^{-\frac{1}{2}g_1 \pm \rho_n \cos\alpha_n} \frac{\cos}{\sin}\left\{n\theta \mp \rho_n \sin\alpha_n \log r\right\}. \quad \ldots \quad (23.8)$$

There is always one of the two alternative forms which tends to zero, and one which tends to infinity, with r. For since

$$\rho_n^2 \cos^2 \alpha_n - g_1^2/4 = n^2 + \rho_n^2 \sin^2 \alpha_n,$$

it follows that $\rho_n \cos \alpha_n$ is always numerically greater than $\frac{1}{2}g_1$.

In the case of the solution which tends to zero when r tends to infinity, the vorticity will decrease according to some (non-integral) power of r. We note that the loci $\zeta = 0$ turn out to be a series of equiangular spirals

$$n\theta \mp \rho_n \sin \alpha_n \log r = s\pi + \text{const.} \quad (s = \text{integer}),$$

giving a periodicity in log r for a given value of θ.

The complete solution for Z_0 will be obtained by adding such solutions, corresponding to $n = 1, 2, 3, \ldots$, etc.

Z_0 will give the vorticity in the case $\kappa = 0$, that is, for an infinitely slow stream or an infinitely high viscosity. The successive terms Z_1, Z_2, etc., obtained from (23.51), etc., will, of course, modify this solution, but it seems possible that a certain periodicity in log r may persist. This appears suggestive in view of the known fact that a sequence of more or less discrete vortices are known to be formed behind such a cylinder, although the motion in the actual case is, of course, not steady.

In any case it is suggested that the functions defined by the differential equation (23.2) should be examined and studied by the pure mathematician, and that this differential equation might eventually replace OSEEN'S equation as the basis of an approximate treatment of the problem of motion of a stream of viscous fluid, slightly disturbed by a cylindrical obstacle.